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Moment problem for majorated operators

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Abstract

We present a moment problem in the context of majorated operators on the space of continuous functions. A generalization of the Hausdorff moment theorem is formulated and proved.
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Let X be a normed space and (x_n) a sequence of linearly independent elements of X. If we have an operator U on X into some space Y we can form a sequence of elements of Y

$$U(x_n) = y_n, \quad n = 0, 1, \ldots \tag{1}$$

A moment problem in a wide sense may be stated as follows: Given a sequence (y_n) of elements of Y find an operator U satisfying (1). The solution of the problem stated in such a wide sense can not be done in a satisfactory manner. Nevertheless even in a general case one can formulate some conditions of solvability and uniqueness of solution of the moment problem.

If the set (x_n) is fundamental, i. e. generating X, an operator U (if it exists) is uniquely determined by the sequence (y_n). It is easy to see that fundamentality of (x_n) is also a necessary condition of a unique solution of the moment problem. Of course we need some kind of linearity in the space Y. We shall suppose that Y is a normed vector space. If we denote by U an operator (not necessarily additive) defined on a set (x_n) by

$$U(x_n) = y_n, \quad n = 0, 1, \ldots \tag{2}$$

then, as existence of an operator U is equivalent to possibility of a linear extension of U to a linear hull $co((x_n))$, a necessary and sufficient condition for solving moment problem there is existence of such a constant M that for any complex numbers $c_0, c_1, \ldots,$ the inequality

$$\left\|\sum c_k y_k\right\| \leq M||c_k x_k||_X \tag{3}$$

is satisfied ($||.||$ denotes a norm on Y and $||.||$ a norm on X) or equivalently

$$\sup \frac{||\sum c_k y_k||}{||c_k x_k||_X} < M, \tag{4}$$

the supremum being taken over all possible

$$c_0, c_1, \ldots, c_n, \quad (n = 0, 1, \ldots).$$

We shall consider only a power moment problem when $X = C[a, b]$ and

$$x_n(t) = t^n, \quad n = 0, 1, \ldots \tag{5}$$

Recall that the function $g(t) : [a, b] \to Y$ is said to have bounded variation $|g|$ if $\sup \sum_j ||g(s_j) - g(s_{j-1}|| < \infty$ where all possible partitions od $[a, b]$ are considered. We can formulate a task in the considered case as follows: Decide under which conditions there exists such a function of bounded variation $g(t) : [a, b] \to Y$ that

$$\int_a^b t^n dg(t) = y_n, \quad n = 0, 1, \ldots \tag{6}$$

The conditions (3) and (4) of solvability of the moment problem (6) can be written with t^k instead of x_k.

We shall derive a concrete result relating to a power moment problem in the interval $[0, 1]$.

Theorem 1 *In order that there exists a function of bounded variation $g(t) : [0, 1] \to Y$ such that*

$$\int_0^1 t^n dg(t) = y_n, y_n \in Y, n = 0, 1, \ldots \tag{7}$$

it is necessary and sufficient that there exists a constant M such that

$$\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} ||\Delta^{n-k} y_k|| \leq M, n = 0, 1, \ldots, \tag{8}$$

where $\Delta^m y_k$ denotes the m-th differences for the sequence (y_k) defined inductively by equalities

$$\begin{gathered} \Delta^{m+1} y_k = \Delta^m y_k - \Delta^m y_{k+1}, \quad \Delta^0 y_k = y_k, \\ m = 0, 1, \ldots; k = 0, 1, \ldots. \end{gathered} \tag{9}$$

Proof. The necessity. Let the moment problem (8) be solved. Denote by L a bounded linear operator on the space $C[0, 1]$ generated by a function of bounded variation, $g(t) : [0, 1] \to Y$,

$$L(f) = \int_0^1 f(t) dg(t), \quad f \in C[0, 1]$$

Put

$$x_k^{(m)}(t) = t^k (1 - t)^m, \quad m, k = 0, 1, \ldots. \tag{10}$$

Since

$$\begin{gathered} x_k^{(m+1)}(t) = t^k(1-t)^{m+1} = t^k(1-t)^m - t^{k+1}(1-t)^m = \\ x_k^{(m)}(t) - x_{k+1}^{(m)}(t), \end{gathered}$$

we have

$$L(x_k^{(m+1)}) = L(x_k^{(m)}) - L(x_{k+1}^{(m)}), m, k = 0, 1, \ldots.$$

Further

$$L(x_k^{(0)}) = y_k.$$

If we take into the consideration (10), we can easily see (by induction) that

$$L(x_k^{(m)}) = \Delta^m y_k, \quad m, k = 0, 1, \ldots.$$

From this we deduce that (8) is satisfied. Indeed,

$$\begin{aligned} \sum_{k=0}^n \begin{pmatrix} n \\ k \end{pmatrix} ||\Delta^{n-k} y_k|| &= \sum_{k=0}^n \begin{pmatrix} n \\ k \end{pmatrix} || \int_0^1 x_k^{(n-k)}(t) dg(t)|| \\ &\leq \sum_{k=0}^n \begin{pmatrix} n \\ k \end{pmatrix} \int_0^1 |x_k^{n-k)}(t)| d|g|(t) \\ &= \int_0^1 \left(\sum_{k=0}^n \begin{pmatrix} n \\ k \end{pmatrix} t^k (1-t)^{n-k} \right) d|g|(t) \\ &= \int_0^1 [t + (1-t)]^n d|g|(t) \leq |g|[0, 1], \end{aligned}$$

where $n = 1, 2, \ldots$. The necessity of the condition is proved, if we put $M = |g|[0,1]$.

The sufficiency. Let L_0 denote the operator defined on the set of functions (x_n), $x_n(t) = t^n$, $\quad n = 0, 1, \ldots$ into Y by formula $L_0(x_n) = y_n$, $\quad n = 0, 1, \ldots$. Extend L to the linear hull of the set (x_n), i. e. to the set of all polynomials. Namely if $x(t) = c_0 + c_1 t + \ldots + c_n t^n$, we put

$$L(x) = c_0 y_0 + c_1 a_1 = \ldots + c_n y_n.$$

Since the functions x_n are linearly independent the definition of L will be unique.

The operator L defined above will be clearly additive and homogeneous. We shall see that condition (9) implies that L is bounded.

Let us note that (even without this condition) the operator L is bounded on the set P_m of polynomials the degree of which is $\leq m$, because P_m is finite-dimensional space (as coordinates we take coefficients of polynomial), hence the convergence in P_m is coordinatewise.

We have

$$L(x_k^{(s)}) = \Delta y_k^s, \quad s, k = 0, 1, \ldots.$$

Take any polynomial $p(t)$. Let the degree of $p(t)$ be m. Form the sequence of corresponding Bernstein polynomials (of $p(t)$)

$$p_n(t) = B_n(p;t) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) p\left(\frac{k}{n}\right) t^k (1-t)^{n-k}.$$

It is well-known that the degree of the polynomial $p_n(t)$ for any $n = 1, 2, \ldots$ is not greater than m, and since $p_n(t)$ uniformly converges to $p(t)$ for $n \to \infty$, we have (according to remarks above) $L(p_n) \to L(p)$.

To obtain our required function, we use Lemma (see [1], §19, p. 380, 383). For each subset A of $[a, b]$, let $C([a, b], A)$ denote the space of continuous functions on $[a, b]$ vanishing outside A. If $F : C([a, b]) \to X$ is a linear mapping, define for each A,

$$|||F_A||| = \sup \sum ||F(\psi_i)||$$

where supremum is over all finite families ψ_i in $C([a, b], A)$ with $\sum |\psi_i| \leq \chi_A(t)$ for all t in $[a, b]$.

□

Lemma 1 *If $F : C([a, b]) \to X$ is a linear mapping, then there exists a (regular) Borel measure $\mu : \mathcal{B}([a, b]) \to X$ with finite variation such that*

$$F(\psi) = \int_a^b \psi(t) d\mu(t),\ \psi \in C([a, b]),$$

if and only if

$$|||F_A||| < \infty$$

for all A in $\mathcal{B}([a, b])$. A mapping F with such a property is called majorated (also dominated, ([1]).

An equivalent definition is as follows.

If $F : C([a, b]) \to X$ is a linear mapping, it is majorated (dominated) if and only if there exists a nonnegative Borel measure μ on $B[0, 1]$ such that

$$||F(\psi)|| \leq \int_a^b |\psi(t)| d\mu(t),\ \psi \in C([a, b]),$$

To prove that

$$|||L_A||| = \sup \sum ||L(\psi_i)||$$

is finite, we take for every ψ_i corresponding Bernstein polynomial

$$p_n^i(t) = B_n(\psi_i; t) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \psi_i \left(\frac{k}{n}\right) t^k (1-t)^{n-k}.$$

Then we have

$$\sum ||L(p_n^i)|| \leq \left(\sum \left|\psi_i \left(\frac{k}{n}\right)\right|\right) \cdot \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) ||\Delta^{n-k} y_k|| \leq M, n = 0, 1, \ldots.$$

We have the equivalent form of the moment theorem:

Theorem 2 *In order that there exists a vector measure of bounded variation $\mu : B[0,1] \to Y$ such that*

$$\int_0^1 t^n d\mu(t) = y_n, y_n \in Y, n = 0, 1, \ldots, \tag{11}$$

it is necessary and sufficient that there exists a constant M such that

$$\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) ||\Delta^{n-k} y_k|| \leq M \quad (n = 0, 1, \ldots). \tag{12}$$

Remark 1 *For suitable auxiliary readings, we refer to [2] and [3].*

References

[1] DINCULEANU, N.: Vector measures. VEB, Berlin, 1966.

[2] HAUSDORFF, F.: Momentprobleme für ein endliches Interval. Math. Z. 16(1923), 220–248.

[3] WIDDER, D. V.: The Laplace transform. Princeton University Press, Princeton, 1946.