SEARCHING THE FRONTIER OF THE PYTHAGOREAN SYSTEM

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Abstract

It is known that the perfect fifth and octave can be expressed as \( X^4Y^3 \) and \( X^7Y^5 \) respectively, where \( X = \frac{256}{243} = 2^{\frac{8}{3}} \) is the minor and \( Y = \frac{9}{8} : X = 3^{\frac{7}{2}} \) the major Pythagorean semitone. In this paper, Pythagorean system is studied when couples of semitones need not be considered in rational numbers; we deal with semitone couples \((X,Y)\) in algebraic numbers expressed with \(d\)th roots, \(d = 1, 2, 3, \ldots\).

Keywords. Pythagorean system, Many valued (fuzzy) coding of musical information

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1 Anatomy of the whole Pythagorean tone

The Pythagorean system was developed in antiquity and satisfactorily formalized by Boethius (480 – 524). Musically, this system is based on the perfect fifths (or, equivalently, on the perfect fourths) and octaves.

Examples. Usually, under Pythagorean Tuning (Scale) (cf. e.g. [1], [5], [2]), we mean one of the following number sequences (physically: the relative frequencies of tones within the octave \([1/1; 2/1]\) ): \(P_2, P_3, P_5, P_7, P_{12}, P_{17}, P_{22}, P_{27}, P_{31} \subset \mathcal{P} = \{2^{\alpha}3^{\beta}; \alpha, \beta \in \mathbb{Z}\}, \mathbb{Z} = \{-1, 0, 1, 2, \cdots\}\)

\[
\begin{align*}
P_2 &= (1/1, 3/2), \\
P_3 &= (1/1, 4/3, 3/2), \\
P_5 &= (1/1, 9/8, 81/64, 3/2, 27/16), \\
P_7 &= (1/1, 9/8, 81/64, 4/3, 3/2, 27/16, 16/9), \\
P_{12} &= (1/1, 2 187/2 048, 9/8, 32/27, 81/64, 4/3, 729/512, 3/2, 6 561/4 096, 27/16, 16/9, 243/128), \\
P_{17} &= (1/1, 256/243, 2 187/2 048, 9/8, 32/27, 19 683/16 384, 81/64, 4/3, 1 024/729, 729/512, 3/2, 128/81, 6 561/4 096, 27/16, 16/9, 59 049/32 768, 4 096/2 187, 243/128), \\
P_{22} &= (1/1, 531 441/524 288, 256/243, 2 187/2 048, 65 536/59 049, 9/8, 32/27, 19 683/16 384, 8 192/6 561, 81/64, 4/3, 177 147/131 072, 1 024/729, 729/512, 3/2, 128/81, 6 561/4 096, 32 768/19 683, 27/16, 16/9, 59 049/32 768, 4 096/2 187, 243/128), \\
P_{27} &= (1/1, 531 441/524 288, 256/243, 2 187/2 048, 65 536/59 049, 9/8, 4 782 969/4 194 304, 32/27, 19 683/16 384, 8 192/6 561, 81/64, 4/3, 177 147/131 072, 1 024/729, 729/512, 262 144/177 147, 1}
\end{align*}
\]
We will deal with the usual multiplicative group \( \mathcal{L} \), \((\{0, \infty\}, \cdot, 1, \leq)\), equipped with the usual order. In Figure 1, we see the Boethius scheme, skeleton of the Pythagorean whole tone, cf. [5]. If we denote by \( Q = 4/3 \) the perfect fourth and by \( O = 2 \) the octave, then \( R = O/Q = 3/2 \) is the perfect fifth, \( T = R/Q = 9/8 \) the Pythagorean whole tone, lindma; \( V = T/U = 2 187/2 048 \) the major Pythagorean semitone, apotome; \( Z = \sqrt{U} = 3/2 \) diacolma, \( K = V/U = 531 441/524 288 \) comat; \( W = \sqrt{K} = \sqrt{531 \, 441/524 \, 288} \ schizma. We see that the Boethius music theory does not avoid irrational numbers and it can be considered as a microtonal theory in the present modern sense. \( \) Recall that one octave is of the length 1 200 cents \((x \mapsto 1 \log_2 x, 1 \leq x \leq 2)\). So, \( Z \approx 1.026400479 \approx 45.11 \) cents, \( K \approx 1.013643265 \approx 23.46 \) cents; \( W \approx 1.006798522 \approx 11.73 \) cents. For comparison: 81/\(80 = 1.0125 \approx 21.51 \) cents.

By a generalized sequence (net) with values in \( \mathcal{L} \) we mean a function from \( I \) to \( \mathcal{L} \), where \( I \) is a directed partially ordered set. For the generalized sequences in general, cf. [3].

The semitone names \( U \) and \( V \) are unique rational numbers which enables to express Pythagorean Tuning \( P_{31} \) (and also, \( P_{22}, P_{23}, P_{28}; \) for \( P_{7}, \) cf. [1], [2], [4]) via the geometrical generalized sequence (the abbreviation: GGS) in rational numbers with some natural conditions. More precisely, in the form

\[
\{ \Gamma_i = \alpha \cdot X^{m_i}Y^{n_i} ; i \in \mathbb{Z} \}
\]

where \( X, Y \) are positive rational, \( \alpha \) positive real, and (possible not unique for every \( i \in \mathbb{Z} \) \( m_i, n_i \in I \subset \mathbb{Z}^2 \) such that

\[
m_i + n_i = i, \ m_i \leq m_{i+1}, n_i \leq n_{i+1}, \ i \in \mathbb{Z}.
\]

The additional conditions are

\[
\alpha = 1,
\]

\[
X^{m_{12}}Y^{n_{12}} = 2,
\]
\[ X^{m_i}Y^{n_i} = \frac{3}{2}, \quad (5) \]
\[ \Gamma_{j+12k} = 2^k \cdot \Gamma_j, \quad j = 0, 1, 2, \ldots, 11, \quad k \in \mathbb{Z}. \quad (6) \]

The condition (3) says that we consider no shift of the fundamental tone. The condition (4) shows that the 12th degree is the octave. The condition (5) asserts that the seventh degree is the perfect fifth \((3/2)\). The condition (6) describes the so called octave equivalency.

Twelve qualitative music degrees are the characteristic attribute of the European tone systems and culture. They are: the unison (octave), minor second, major second, minor third, major third, fourth, tritone, fifth, minor sixth, major sixth, minor seventh, major seventh. Each degree may contain one or more tuning values. The GGSs \(P_{17}, P_{22}, P_{28}, P_{31}\) have 17, 22, 28, 31 values for 12 degrees within the octave \([0, 2)\), respectively. There are different index sets \(I\) for \(P_{12}, P_{17}, P_{22}, P_{28}, P_{31}\). In this paper, we will consider GGSs only for pairs \((X, Y)\) (two different semitones, not three or more). Examples of index sets \(I\) for GGGs we can see also in Figure 2 (Table 3; a sequence) and in Figure 2 (Table 4). We deal with index sets directed both forward and backward (the two side GGS).

Pythagorean Tunings \(P_{12}, P_{17}, P_{22}, P_{28}, P_{31}\) are GGSs, the solutions of the equation system (1), (2), (3), (4), (5) with rational \((X, Y)\). Therefore, there arises a question: does there exist any GGS, the solution of the equation system (1), (2), (3), (4), (5) with irrational \((X, Y)\)? In this paper, we solve this problem in the positive for semitone couples \((X, Y)\) expressed with algebraic numbers.

The Pythagorean tunings \(P_{17}, P_{22}, P_{28}, P_{31}\) are many valued (fuzzy) coding systems of information in music. For more explanations about the philosophy of fuzziness of tunings in music see [2]. The important and obvious fact is that the theoretical tunings (and operations among them) never occur in the real live music because there are ever at least two different tone systems in the sound interaction during the performance: the tone subset \(A\) of the tuned system itself, and physically, the systems of overtones \(B\) of tones in \(A\). Clearly, always \(A \neq B\).

2 The second level: algebraic numbers

Observe, [1], that \(P_i, i = 2, 3, 5, 7, 12, 17, 22, 28, 31\), can be obtained as the solution of the following Problem 1 and using the enlargement algorithm based on the fifth \(X^{m_i}Y^{n_i} = \frac{3}{2}\) and the octave \(X^{m_{12}}Y^{n_{12}} = 2\), i.e.:
\[ Q_{[k_1, k_2]} = \{(3/2)^k \ (\text{mod} \ 2); k \in [k_1, k_2], \ k_1 < k_2, \ k_1, k_2 \in \mathbb{Z}\}. \]

For instance, \(k = 1, 2, 3, 4, 5\) for \(P_5\).

**Problem 1.**
To find \(X, Y\) rational numbers such that (3), (4), (5) for some \(m_i, n_i\) nonnegative integer numbers such that
\[ m_i + n_i = i, \quad i = 0, 7, 12, \quad 0 \leq m_7 \leq m_{12}, \quad 0 \leq n_7 \leq n_{12}. \]

**Problem 2.** To find all triplets \((m_{12}, m_7, d)\) in nonnegative integers such that
\[ 7m_{12} - 12m_7 = d, \quad 0 \leq 7 - m_7 \leq 12 - m_{12}, \quad 0 \leq m_7 \leq m_{12}. \]
\[ 7m_{12} - 12m_7 = d \]

Table 1: Solutions of \( 7m_{12} - 12m_7 = d \)

<table>
<thead>
<tr>
<th>( m_7 )</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>7,14,21,28,35, 42,49,56,63,70,77</td>
</tr>
<tr>
<td>1</td>
<td>2,9,16,23,30, 37,44,51,58,65</td>
</tr>
<tr>
<td>2</td>
<td>4,11,18,25, 32,39,46,53</td>
</tr>
<tr>
<td>3</td>
<td>6,13,20, 27,34,41</td>
</tr>
<tr>
<td>4</td>
<td>1,8,15,22,29</td>
</tr>
<tr>
<td>5</td>
<td>3,10,17</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
</tr>
</tbody>
</table>

The solution of Problem 1 is given by the solution of Problem 2. Indeed, for \( d = 0 \), we have \( m_{12} = 12, m_7 = 7 \), i.e. Equal Temperament (\( X = Y = \sqrt[12]{2} \)) which does not contain any perfect fifth 3/2, (analogously for \( d = 12 \)). If \( d \neq 0 \) and \((m_{12}, m_7, d)\) is a solution of Problem 2, then the solution of (3), (4), (5) we obtain as follows:

\[
X = 2^{E_2 \cdot X} 3^{E_3 \cdot X}, \quad Y = 2^{E_2 \cdot Y} 3^{E_3 \cdot Y},
\]

where \( E_{2, X} = (19 - m_7 - m_{12})/d \), \( E_{3, X} = (m_{12} - 12)/d \), \( E_{2, Y} = (-m_{12} - m_7)/d \), \( E_{3, Y} = m_{12}/d \). The numbers \( X, Y \) are rational solutions of Problem 1 if and only if \( d = 1 \) (or \( d = -1 \), the symmetrical case). For \( d = 1 \), the values are \( X = U, Y = V \).

Now it is possible to extend Pythagorean Tuning (in any form: \( P_2 \) or \( \ldots \) or \( P_3 \)) to algebraic numbers: to solve Problem 1a, where

**Problem 1a**

To find \( X, Y \) algebraic numbers such that (4), (5) for some \( m_i, n_i \) nonnegative integer numbers such that

\[ m_i + n_i = i, \quad i = 0, 7, 12, \quad 0 \leq m_7 \leq m_{12}, 0 \leq n_7 \leq n_{12}. \]

**Theorem 1** For every \( d \in \mathbb{Z} \), there exists unique or none solution \((X_d, Y_d)\) of Problem 1a such that \( X_d, Y_d \) are expressed via \( d \)th roots, multiplication, and/or dividing of numbers 2 and 3.

**Proof** The solutions of the Diophantine equation

\[ 7m_{12} - 12m_7 = d, \quad 0 \leq m_7 \leq m_{12}, \]

where \( d, m_7, m_{12} = 0, 1, 2, 3, 4 \ldots \) are collected in Table 1. By (2), we obtained values \( n_{12}, n_7 \). Take only the solutions such that \( n_{12} \geq n_7 \). Then the solutions of Problem 2 are collected in Table 2. By (7) we obtain the solutions of Problem 1a (\( d < 0 \) are the symmetrical cases).

**Theorem 2** Denote by \( G_d \) the union of all GGSs as sets such that they are generated by the solution \((X_d, Y_d)\) of Problem 1a, \( d \) in Table 2. Then

\[
\begin{align*}
G_1 &= P = \{2^\alpha 3^\beta; \alpha, \beta \in \mathbb{Z}\}, \\
G_2 &= P \cup (\sqrt[12]{2}) \cdot P \\
G_3 &= P \cup (\sqrt[4]{3}) \cdot P \cup (3\sqrt[12]{2}) \cdot P \\
&\quad \ldots \\
G_d &= \bigcup_{i=1}^{d} \beta_i \cdot P \\
&\quad \ldots \\
G_{35} &= \bigcup_{i=1}^{35} \beta_i \cdot P 
\end{align*}
\]
Then where $d$

Consider the set $12 \in \mathbb{Z}$.

Clearly, $Q_{\mathbb{Z}} = \mathcal{P}$ and $Q_{\mathbb{Z}} \cup Q_{\mathbb{Z}^\prime} = (3/2)^k \mod 2$ for every $k_1, k_2 \in \mathbb{Z}$, $k_1 < k_2$.

The assertion $G_1 = \mathcal{P}$ is obvious for $d = 1$, $X_1 = X = 256/243, Y_1 = Y = (9/8): X_1$.

For $d = 2$, $X_2 = 256/243, Y_2 = 3\sqrt{3}$. We have (for illustration, cf. Figure 2):

$$Q_{\mathbb{Z}} = \mathcal{P} = \bigcup_{p \in \mathbb{Z}} \bigcup_{q \in \mathbb{Z}} X_2^p Y_2^{2q}.$$  

Then

$$\mathcal{G}_2 = \mathcal{P} \cup (Y_2) \cdot \mathcal{P} = \mathcal{P} \cup (\sqrt{2}) \cdot \mathcal{P}.$$  

For $d = 3$, $X_3 = \sqrt{32}/3, Y_3 = 27/\sqrt{16384}$. We have (for illustration, cf. Figure 3):

$$Q_{\mathbb{Z}} = \mathcal{P} = \bigcup_{p \in \mathbb{Z}} \bigcup_{q \in \mathbb{Z}} X_3^p Y_3^q \cdot X_3^{3q}.$$  

Then

$$\mathcal{G}_3 = \mathcal{P} \cup (X_3)\cdot \mathcal{P} \cup (X_3^2)\cdot \mathcal{P} = \mathcal{P} \cup (\sqrt{3})\mathcal{P} \cup (3\sqrt{2})\cdot \mathcal{P}.$$  

Table 2: Solutions of Problem 2

<table>
<thead>
<tr>
<th>C</th>
<th>$X_{2Y_2^p}$</th>
<th>0.00</th>
<th>$U^dK^p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_2$</td>
<td>$X_{2Y_2^3}$</td>
<td>3/1</td>
<td>$U^1K^{1/2}$</td>
</tr>
<tr>
<td>$D$</td>
<td>$X_{2Y_2^2}$</td>
<td>9/8</td>
<td>$U^2K^1$</td>
</tr>
<tr>
<td>$E_2$</td>
<td>$X_{2Y_2^3}$</td>
<td>27/3</td>
<td>$U^3K^{3/2}$</td>
</tr>
<tr>
<td>$E$</td>
<td>$X_{2Y_2^3}$</td>
<td>81/64</td>
<td>$U^4K^2$</td>
</tr>
<tr>
<td>$F_2$</td>
<td>$X_{2Y_2^3}$</td>
<td>4/3</td>
<td>$U^5K^3$</td>
</tr>
<tr>
<td>$G$</td>
<td>$X_{2Y_2^2}$</td>
<td>3/2</td>
<td>$U^6K^3$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$X_{2Y_2^2}$</td>
<td>9/32</td>
<td>$U^8K^{7/2}$</td>
</tr>
<tr>
<td>$A$</td>
<td>$X_{2Y_2^2}$</td>
<td>27/16</td>
<td>$U^9K^4$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$X_{2Y_2^p}$</td>
<td>81/2048</td>
<td>1007.82</td>
</tr>
<tr>
<td>$B$</td>
<td>$X_{2Y_2^p}$</td>
<td>243/128</td>
<td>1007.82</td>
</tr>
</tbody>
</table>

Table 3: $d = 2$, $P_7$ (○), $(\sqrt{2}) \cdot P_3$ (●)

where $\beta_1, \ldots, \beta_d$ are pairwise incommensurable algebraic numbers for every $d$.

**Proof**

Consider the set

$$Q_{\mathbb{Z}} = \{(3/2)^k \mod 2; k \in \mathbb{Z}\} = \{(X^{m_1}Y^{n_1})^k \mod X^{m_2}Y^{n_2}; k \in \mathbb{Z}\}.$$  

For $d = 2$, $X_2 = 256/243, Y_2 = 3\sqrt{3}$. We have (for illustration, cf. Figure 2):

$$Q_{\mathbb{Z}} = \mathcal{P} = \bigcup_{p \in \mathbb{Z}} \bigcup_{q \in \mathbb{Z}} X_2^p Y_2^{2q}.$$  

Then

$$\mathcal{G}_2 = \mathcal{P} \cup (Y_2) \cdot \mathcal{P} = \mathcal{P} \cup (\sqrt{2}) \cdot \mathcal{P}.$$  

For $d = 3$, $X_3 = \sqrt{32}/3, Y_3 = 27/\sqrt{16384}$. We have (for illustration, cf. Figure 3):

$$Q_{\mathbb{Z}} = \mathcal{P} = \bigcup_{p \in \mathbb{Z}} \bigcup_{q \in \mathbb{Z}} X_3^p Y_3^q \cdot X_3^{3q}.$$  

Then

$$\mathcal{G}_3 = \mathcal{P} \cup (X_3)\cdot \mathcal{P} \cup (X_3^2)\cdot \mathcal{P} = \mathcal{P} \cup (\sqrt{3})\mathcal{P} \cup (3\sqrt{2})\cdot \mathcal{P}.$$  

5
<table>
<thead>
<tr>
<th>$C$</th>
<th>$X Y Z$</th>
<th>$d$</th>
<th>$P_5$</th>
<th>$U^2 K^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_0$</td>
<td>$X Y Z$</td>
<td>$\sqrt[3]{32}/3$</td>
<td>98.04</td>
<td>$U^4 K^{10/3}$</td>
</tr>
<tr>
<td>$C_1$</td>
<td>$X Y Z$</td>
<td>$27/\sqrt{168}$</td>
<td>105.86</td>
<td>$U^3 K^{2/3}$</td>
</tr>
<tr>
<td>$D$</td>
<td>$X Y Z$</td>
<td>$9/8$</td>
<td>203.91</td>
<td>$U^4 K^4$</td>
</tr>
<tr>
<td>$E_0$</td>
<td>$X Y Z$</td>
<td>$3/\sqrt{16}$</td>
<td>301.96</td>
<td>$U^4 K^{4/3}$</td>
</tr>
<tr>
<td>$D_2$</td>
<td>$X Y Z$</td>
<td>$243/\sqrt{3388608}$</td>
<td>309.78</td>
<td>$U^3 K^{5/3}$</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$X Y Z$</td>
<td>$81/64$</td>
<td>407.82</td>
<td>$U^4 K^4$</td>
</tr>
<tr>
<td>$F_0$</td>
<td>$X Y Z$</td>
<td>$9/\sqrt{256}$</td>
<td>505.86</td>
<td>$U^3 K^{1/3}$</td>
</tr>
<tr>
<td>$G$</td>
<td>$X Y Z$</td>
<td>$3/2$</td>
<td>603.91</td>
<td>$U^4 K^{2/3}$</td>
</tr>
<tr>
<td>$A_0$</td>
<td>$X Y Z$</td>
<td>$\sqrt[3]{4}$</td>
<td>800.00</td>
<td>$U^8 K^{10/3}$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$X Y Z$</td>
<td>$81/\sqrt{3388608}$</td>
<td>807.82</td>
<td>$U^8 K^{11/3}$</td>
</tr>
<tr>
<td>$A$</td>
<td>$X Y Z$</td>
<td>$27/16$</td>
<td>905.86</td>
<td>$U^4 K^4$</td>
</tr>
<tr>
<td>$B_0$</td>
<td>$X Y Z$</td>
<td>$9/\sqrt{128}$</td>
<td>1003.91</td>
<td>$U^{10} K^{3/3}$</td>
</tr>
<tr>
<td>$B$</td>
<td>$X Y Z$</td>
<td>$3/\sqrt{4}$</td>
<td>1101.96</td>
<td>$U^{10} K^{3/3}$</td>
</tr>
<tr>
<td>$C^*$</td>
<td>$X Y Z$</td>
<td>$2$</td>
<td>1200.00</td>
<td>$U^{12} K^3$</td>
</tr>
</tbody>
</table>

Table 4: $d = 3$, $P_5$ (○), $\left(\sqrt[3]{32}/3\right) \cdot P_5$ (●), $\left(3/\sqrt{4}\right) \cdot P_5$ (■)

Figure 2: $G_2 = P \cup (\sqrt{2}) \cdot P$
Analogously for every $d = 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 18, 20, 21, 23, 25, 28, 30, \text{ and } 35$.

3 Semitone metric space

Now we show that Pythagorean tone systems expressed via GGSs are the closer in some sense to Equal Temperament the greater the positive integer $d$ is.

**Theorem 3** For $K = V/U$ (comat), denote by $\rho(X, Y) = |\log_K(X) - \log_K(Y)|$, $X, Y \in \mathcal{L}$. Let $(X_{d_1}, Y_{d_1}), (X_{d_2}, Y_{d_2})$ are solutions of Problem 1a corresponding $d_1, d_2$ and $d_1 < d_2$, $d_1, d_2 = 1, 2, \ldots$. Then $\rho(X_1, Y_1) > \rho(X_2, Y_2)$.

**Proof.** It can be verified directly that $(\mathcal{L}, \rho)$ is a metric space.

If $(X_d, Y_d)$ is a couple of numbers which is solution of Problem 1a corresponding $d, d = 0, 1, 2, 3, \ldots$, then by (7),

$$|\log_K(Y_d) - \log_K(X_d)| = \log_K 2 - \frac{m_{12} - m_7}{3} - \log_K 2 - \frac{m_{12} - m_7}{3} - \frac{m_9 - 12}{3} \log_K 3 - \frac{19}{d} \log_K 2 = \frac{1}{d} \log_K 2 - \frac{19}{d} \log_K 2 = \frac{1}{d} \log_K K = \frac{1}{d}.$$

If $d_1 < d_2$, then

$$\rho(X_{d_1}, Y_{d_1}) = \frac{1}{d_1} > \frac{1}{d_2} = \rho(X_{d_2}, Y_{d_2}).$$

**Corollary 1** If $Y_d > X_d$, then $Y_d/X_d = \sqrt[12]{K}$. 

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Figure 3: $\mathcal{G}_3 = P \cup (\sqrt[d]{4}) \cdot P \cup (3\sqrt[12]{2}) \cdot P$
Definition Denote by $S$ the set of all numbers $X \in \mathcal{L}$ such that $(X,Y)$ is a solution of Problem 1a for some integer $m_7, m_{12}, d$ and $Y \in \mathcal{L}$. We will call the metric space $(S, \rho)$ the metric space of semitones generating the perfect fifths.

4 Applications

The sensitivity of human perception apparatus is 5 – 6 cents. The psycho-acoustical boundary (to what the listener or player tent to) is 2 – 3 cents. This is the reason why there is no practical sense to use GGSs for $d > 12$, we hear no difference between Equal Temperament and the constructed GGS (we do not suppose too many values within 12 qualitative degrees). From the acoustical point of view, the interesting are cases of small values of $d = 1, 2, 3, \ldots, 11$ for which $Y_d/X_d \approx 23.46, 11.73, 7.82, 5.86, 4.69, 3.91, 3.35, \ldots, 2.13$ cents, respectively. From the viewpoint of modulations, the ordering is reverse: $d = 11, 10, \ldots, 3, 2, 1$. So, we have to choose an appropriate compromise of $d$ depending on the kind of modulations within a given musical composition (set of compositions). Some of GGSs in this paper, new from the musicology viewpoint, may be considered for tuning of musical instruments.

For $d = 2$, cf. Table 3 (Figure 2). The set $P_7$ is the Pythagorean heptatonic $F \rightarrow C \rightarrow G \rightarrow D \rightarrow A \rightarrow E \rightarrow B$, the “white keys”). The set $(\sqrt{2}) \cdot P_5$, is a Pythagorean pentatonic $F_2 \rightarrow C_2 \rightarrow G_2 \rightarrow D_2 \rightarrow A_2$, (the “black keys”).

For $d = 3$, cf. Table 4 (Figure 3, many valued tuning). The sets $P_5$ (i.e. $C \rightarrow G \rightarrow D \rightarrow A \rightarrow E$), $(3/\sqrt{3}) \cdot P_5$, (i.e. $B \rightarrow F_1 \rightarrow C_2 \rightarrow G_2 \rightarrow D_2$), and $(\sqrt{32}/3) \cdot P_5$ (i.e. $D_b \rightarrow A_b \rightarrow E_b \rightarrow B_b \rightarrow F$) are three Pythagorean pentatonics.

References