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COMMA 32 805/32 768

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We find a fifth approximation of the Just Intonation which generalizes Equal Temperament. The intervals causing a dilemma are the second and the minor seventh and the tritone because they are unambiguous in Just Intonation (the relative frequencies 10/9, 9/8, 8/7 and 7/4, 16/9, 18/10 and 45/32, 64/45, respectively). If we do not consider the second and seventh with the relative frequencies 8/7 and 7/4, respectively, all the music intervals in this approximation either coincide with he Just Intonation interval values (the octave, fifth, fourth, second (9/8) and the minor seventh (16/9)) or are exactly the one comma distant from the corresponding Just Intonation intervals. This comma is 32 805/32 768 $\approx$ 1.00112915, which is less than the ratio of frequencies of the perfect and the equal tempered fifths ($\approx$ 1.00112989).

Keywords: Many-valued coding; Equal temperament; Just Intonation; Pythagorean Tuning; Musical acoustics; Perception; Microtonal music.

1. Introduction

Mathematicians have solved problems connected with tuning since antiquity. It seems that in these considerations various mathematical structures on real line can be applied. Connections between mathematics and music are mutually enriching, depending on the progress in mathematics or music. Therefore problems having their origin in music attract the interest of scientists and musicians throughout the history up to present days.

Pythagorean Tuning, cf. e.g. [1], [2], was created as a sequence of ratios, i.e. products $2^p 3^q$, where p and q are integers, cf. Table 1. This tuning was established about five hundred years B. C. and used in the Western music up to the 14th century.

The gradual development of polyphony led to the introduction of thirds (5/4 and 6/5) and sixths (8/5 and 5/3). After that the problem why musical ratios of integers are considered attempts to provide a strict mathematical definition of consonance. The practical result of this period was the creation of the *Just Intonation* set (abbreviation: JI). JI is based on the intervals which are integer exponents of numbers 2, 3, 5, and 7. The key figures in these studies were G i o s e f f o Z a r l i n o

(1517 – 1590), S i m o n S t e v i n (1548 – 1620), J o h a n n K e p l e r (1571 – 1630). The present microtonalists assert that "Just Intonation is any system of tuning in which all of the intervals can be represented by ratios of whole numbers, with a strongly implied preference for the smallest numbers compatible with a given musical purpose" (David B. Doty). We will deal with JI given by Table 2 (see [3]).

The further progress in tuning continued by so-called *temperaments* which did not avoid the inharmonic music intervals given by irrational numbers. Rather the most popular today, *Equal Temperament*, was already known to A n d r e a s W e r c k m e i s t e r in 1698. It is represented by the sequence

$$W = \{1, \sqrt[12]{2}, (\sqrt[12]{2})^2, \ldots, (\sqrt[12]{2})^{12} = 2\} = \{C^{(W)}, C_\sharp^{(W)}, \ldots, B^{(W)}, C'^{(W)}\}.$$

Then there have been many theoretical and experimental works concerned with the assumed universality of *diatonic scales* – both *pro* and *contra*, cf. [4], [5]. However, our aim is not to judge such non-mathematical problems.

In the present paper, we find a fifth approximation of the JI, cf. Table 3, which generalizes Equal Temperament. The intervals causing a dilemma are the second and the minor seventh and the tritone because they are unambiguous (the relative frequencies [10/9, 9/8, 8/7] and [7/4, 16/9, 18/10] and [45/32, 64/45]; respectively) in JI. If we do not consider the second and seventh with the relative frequencies 8/7 and 7/4, respectively, all the music intervals in this approximation either coincide with the JI interval values (the octave, fifth, fourth, second (9/8) and the minor seventh (16/9)) or are exactly the one comma distant from the corresponding JI intervals. **This comma is 32 805/32 768** (≈ 1.00112915), **which is less than the ratio of frequencies of the perfect and the equal tempered fifths** (≈ 1.00112989).

The author is indebted the referee for pointing out that related ideas are treated in a more general setting in several papers of Yves Hellegouarch, in particular, in "Gammes naturelles", Publications de l'APMER, n° 53, 1983; so, some other papers of Yves Hellegouarch are quoted in the references below, [6], [7].

2. Preliminaries

Denote by $\mathcal{N}, \mathcal{Z}, \mathcal{Q}, \mathcal{R}, \mathcal{C}$ the sets of all natural, integer, rational, real, and complex numbers, respectively. Denote by $L = ((0,\infty), \cdot, 1, \leq)$ the usual multiplicative group with the natural order on $\mathcal{R}$. So, if $a \leq b$ with $a, b \in (0,\infty)$, then b/a is an L-length of the interval (a,b). Since this terminology is not obvious, we borrow the usual musical terminology, i.e. we simply say that b/a is an interval. This inaccuracy does not lead to any misunderstanding because in this paper the therm "interval" is used only in this sense.

The proof of the following lemma is easy.

Lemma 1. *Let* $q = \text{const} \in (0,\infty)$.
Then $\rho_q(u,v) = \left|\log_2 \sqrt[q]{u/v}\right|$, $u, v \in (0,\infty)$, *is a metric on* L.

Equal Temperament can be generalized as follows:

Table 1. Pythagorean Tuning.

$C^{(P)}$	$2^0 3^0$	1/1	1.0
$D_\flat^{(P)}$	$2^8 3^{-5}$	256/243	1.053497942
$C_\sharp^{(P)}$	$2^{-11} 3^7$	2187/2048	1.067871094
$D^{(P)}$	$2^{-3} 3^2$	9/8	1.125
$E_\flat^{(P)}$	$2^5 3^{-3}$	32/27	1.185185185
$D_\sharp^{(P)}$	$2^{-14} 3^9$	19683/16384	1.201354981
$E^{(P)}$	$2^{-6} 3^4$	81/64	1.265625
$F^{(P)}$	$2^2 3^{-1}$	4/3	1.333333333
$G_\flat^{(P)}$	$2^{10} 3^{-6}$	1024/729	1.404663923
$F_\sharp^{(P)}$	$2^{-9} 3^6$	729/512	1.423828125
$G^{(P)}$	$3^1 2^{-1}$	3/2	1.5
$A_\flat^{(P)}$	$2^7 3^{-4}$	128/81	1.580246914
$G_\sharp^{(P)}$	$2^{-12} 3^8$	6561/4096	1.601806641
$A^{(P)}$	$2^{-4} 3^3$	27/16	1.6875
$B_\flat^{(P)}$	$2^4 3^{-2}$	16/9	1.777777777
$A_\sharp^{(P)}$	$2^{-15} 3^{10}$	59049/32768	1.802032473
$B^{(P)}$	$2^{-7} 3^5$	243/128	1.898437528
$C'^{\ (P)}$	2	2/1	2.0

Table 2. Just Intonation.

$C^{(JI)}$	$2^0 3^0 5^0$	1/1	1.0
$C_\sharp^{(JI)}, D_\flat^{(JI)}$	$2^4 3^{-1} 5^{-1}$	16/15	1.066666666
$D^{(JI)}$	$2^1 3^{-2} 5^1$	10/9	1.111111111
	$2^{-3} 3^2$	9/8	1.125
	$2^3 7^{-1}$	8/7	1.14285714
$D_\sharp^{(JI)}, E_\flat^{(JI)}$	$2^1 3^1 5^{-1}$	6/5	1.2
$E^{(JI)}$	$2^{-2} 5^1$	5/4	1.25
$F^{(JI)}$	$2^2 3^1$	4/3	1.333333333
$F_\sharp^{(JI)}$	$2^{-5} 3^2 5^1$	45/32	1.40625
$G_\flat^{(JI)}$	$2^6 3^{-2} 5^{-1}$	64/45	1.422222222
$G^{(JI)}$	$2^{-1} 3^1$	3/2	1.5
$G_\sharp^{(JI)}, A_\flat^{(JI)}$	$2^3 5^{-1}$	8/5	1.6
$A^{(JI)}$	$3^{-1} 5^1$	5/3	1.666666666
$A_\sharp^{(JI)}, B_\flat^{(JI)}$	$2^{-2} 7^1$	7/4	1.75
	$2^4 3^{-2}$	16/9	1.777777777
	$3^2 5^{-1}$	9/5	1.8
$B^{(JI)}$	$2^{-3} 3^1 5^1$	15/8	1.875
$C'^{(JI)}$	2^1	2/1	2.0

Definition 1. Let $\{x, \ldots, y \in \mathcal{R};\ 1 < x \leq \ldots \leq y < 10/9\}$ be a set of N numbers. Let $m_0, m_1, \ldots, m_M;\ \ldots\ ; n_0, n_1, \ldots, n_M$ be $M \times N$ nonnegative integers, such that

$$0 = m_0 \leq m_1 \leq \ldots \leq m_M;\ \ldots;\ 0 = n_0 \leq n_1 \leq \ldots \leq n_M$$

and

$$m_j + \ldots + n_j = j,\ j = 0, 1, \ldots, M.$$

Then the set $S = \{x^{m_0} \ldots y^{n_0}, x^{m_1} \ldots y^{n_1}, \ldots, x^{m_M} \ldots y^{n_M}\}$ is said to be an *M-degree N-interval scale*. Particularly, a 12-degree N-interval scale S is a *12-degree 2-interval* (2/1, 3/2)*-scale* if

$$x^{m_{12}} y^{n_{12}} = 2/1, x^{m_7} y^{n_7} = 3/2.$$

We say that $\mathcal{S}$ is a *M-degree N-interval system ((2/1, 3/2)-system)*, if $\mathcal{S} = \bigcup S$, where S are M-degree N-interval scales ((2/1, 3/2)-scales).

Equal Tempered Scale is a 12-degree 1-interval scale which is not a 12-degree (2/1, 3/2)-scale. Pythagorean Tuning is a 12-degree 2-interval (2/1, 3/2)-system. The constructed approximation of JI is also a 12-degree 2-interval (2/1, 3/2)-system.

Lemma 2. *Let $x, y, a \in (1; \infty),\ m, n \neq 0,\ m, n \in \mathcal{R}$. Then the following equalities are equivalent: (1) $a = x^m y^n$, (2) $1 = m/\log_x a + n/\log_y a$.*

Proof. $a = x^m y^n \iff \log a = m \log x + n \log y \iff 1 = m\frac{\log x}{\log a} + n\frac{\log y}{\log a} \iff 1 = \frac{m}{\log_x a} + \frac{n}{\log_y a}$. □

As a consequence of Definition 1 we obtain the following lemma, cf. [8].

Lemma 3. *The unique two rational intervals in Definition 1 for 12-degree 2-interval (2/1, 3/2)-scales are*

$$x = 2^8/3^5, y = 3^7/2^{11}.$$

In what follows, fix $\mathbf{x = 2^8/3^5, y = 3^7/2^{11}}$.

Denote by $\mathcal{Q}_{x,y} = \{x^\alpha y^\beta; \alpha, \beta \in \mathcal{Z}\}$ $(\mathcal{R}_{x,y} = \{x^\alpha y^\beta; \alpha, \beta \in \mathcal{R}\})$. Consider the following map $\theta : x^\alpha y^\beta \to (\alpha, \beta)$, $\theta : \mathcal{Q}_{x,y} \to \mathcal{Z}^2$ $(\theta_* : \mathcal{R}_{x,y} \to \mathcal{C})$ and

$$(x^\alpha y^\beta)(x^\gamma y^\delta) \to (\alpha + \gamma, \beta + \delta), (x^\alpha y^\beta)^\gamma \to (\gamma\alpha, \gamma\beta),\ \alpha, \beta, \gamma, \delta \in \mathcal{Z}\ (\mathcal{R}).$$

Lemma 4. *The map θ is an isomorphism.*

Proof. It is enough to show that the map $\theta : \mathcal{Q}_{x,y} \to \mathcal{Z}^2$ is an injection (the other properties of isomorphism are trivial). Suppose that there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{Z}$, such that $\alpha_1 \neq \alpha_2, \beta_1 \neq \beta_2$ implies $x^{\alpha_1} y^{\beta_1} = x^{\alpha_2} y^{\beta_2}$. Then $x^{\alpha_1 - \alpha_2} y^{\beta_1 - \beta_2} = 1$ and $(\alpha_1 - \alpha_2) + (\beta_1 - \beta_2)\log_x y = 0$. This is possible only in the case when $\alpha_1 = \alpha_2, \beta_1 = \beta_2$, a contradiction. Consequently, $\alpha_1 = \alpha_2, \beta_1 = \beta_2$. □

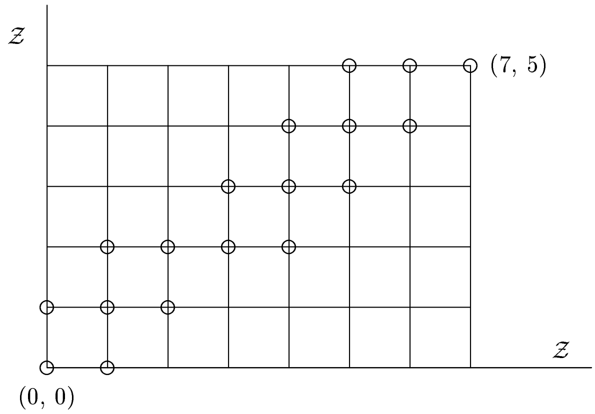

Fig. 1. Image of Pythagorean Tuning in the isomorphism θ.

Embed $\mathcal{Z}^2$ identically into the complex plane $\mathcal{C}$ (for the process of the standard enlargements $\mathcal{Z} \to \mathcal{Q} \to \mathcal{R} \to \mathcal{C}$, cf. e.g. [9]). Denote this injection by η. Denote by θ_*^{-1} the extended map $\theta_*^{-1} : \mathcal{C} \to \mathcal{R}_{x,y}$. So we have the following commutative diagram which defines the embedding $\varphi : \mathcal{Q}_{x,y} \to \mathcal{R}_{x,y}$:

$$\begin{array}{ccc} \mathcal{Q}_{x,y} & \xrightarrow{\theta} & \mathcal{Z}^2 \\ \cap & & \cap \\ \varphi \big\downarrow & & \big\downarrow \eta \\ \mathcal{R}_{x,y} & \xleftarrow{\theta_*^{-1}} & \mathcal{C} \end{array}$$

Lemma 5 (cf [2]).

The image of Pythagorean Tuning under the isomorphism θ is in Fig. 1.

Define the following JI Approximation.

Definition 2. The *JI Approximation* $\mathcal{A}$ is defined by Table 3.

3. The Just Intonation Approximation $\mathcal{A}$

The following theorem is easy to verify, cf. also Fig. 2.

Theorem 1. *Let* $a_j = 2/1$, $15/8$, $9/5$, $7/4$, $16/9$, $5/3$, $8/5$, $3/2$, $64/45$, $45/32$, $4/3$, $5/4$, $6/5$, $8/7$, $9/8$, $10/9$, $16/15$, *where* $j = 12$, 11, 10, 10, 10, 9, 8, 7, 6, 6, 5, 4, 3, 2, 2, 2, 1, *respectively. For* $x = 2^8/3^5, y = 3^7/2^{11}$, *the system of linear equations*

$$\begin{aligned} \tfrac{m_j}{\log_x a_j} + \tfrac{n_j}{\log_y a_j} &= 1, \\ m_j + n_j &= j \end{aligned}$$

has the following integer solutions $(m_{12}, n_{12}) = (7, 5), (m_{10}, n_{10}) = (6, 4), (m_7, n_7) = (4, 3), (m_5, n_5) = (3, 2), (m_2, n_2) = (1, 1)$, *for* $a_j = 2/1, 16/9, 3/2, 4/3, 9/8$, *where* $j = 12, 10, 7, 5, 2$, *respectively.*

Theorem 2. *The JI Approximation* $\mathcal{A}$*, JI, and Equal Temperament are symmetric with respect to the center point* $(7/2, 5/2)$ *in the map* θ_*.

Proof. For the JI Approximation $\mathcal{A}$, see Fig. 3.

Prove the assertion for JI. Observe that

$$m_j = \frac{j \log y - \log a_j}{\log y - \log x}, n_j = \frac{\log a_j - j \log x}{\log y - \log x}$$

is the solution of the system in Theorem 1. Further, $a_{12-j} = 2/a_j, j = 0, 1, \ldots, 12$, cf. Table 2. Since $2 = x^7 y^5$, we have:

$$m_{12-j} = \frac{(12-j)\log y - \log a_{12-j}}{\log y - \log x} = \frac{12 \log y - j \log y - \log(2/a_j)}{\log y - \log x} =$$

$$= \frac{-j \log y + \log a_j}{\log y - \log x} + \frac{\log y^{12} - \log x^7 y^5}{\log y - \log x} = -m_j + \frac{\log(y^{12}/x^7 y^5)}{\log(y/x)} = -m_j + 7.$$

Analogously, $n_{12-j} = -n_j + 5$.

Equal Temperament is also symmetric in the map θ_* with respect to the center point $(7/2, 5/2)$, cf. Fig. 4 and [2].

So, the point $(7/2, 5/2)$ is a common center of symmetry for Equal Temperament, JI, and the JI Approximation $\mathcal{A}$. Note that $x^{7/2} y^{5/2} = (\sqrt[12]{2})^6 = \sqrt{2}$ (the tritone in Equal Temperament). □

Remark 1. Enlarging the scales as usually by the *octave equivalency* (i.e. multiplying the scale values by 2^i, $i \in \mathcal{Z}$) to the whole L, we see that the tritones in Equal Temperament and also octaves (i.e. $2^i\sqrt{2}, 2^i$, respectively) are all common centers of symmetry of these three tone systems.

The proof of the following theorem is implied by Table 3, the construction of the JI Approximation $\mathcal{A}$.

Theorem 3. *The JI Approximation* $\mathcal{A}$ *is a 12-degree 2-interval (2/1, 3/2)-system.*

Table 3. The JI Approximation $\mathcal{A}$, $x = 2^8/3^5, y = 3^7/2^{11}$.

$C^{(A)}$	x^0y^0	2^03^0	1/1	1.0
$C_\sharp^{(A)}, D_\flat^{(A)}$	x^0y^1	$2^{-11}3^7$	2187/2048	1.067871094
	x^2y^0	$2^{16}3^{-10}$	65536/59049	1.10985791
$D^{(A)}$	x^1y^1	$2^{-3}3^2$	9/8	1.125
	x^0y^2	$2^{-22}3^{14}$	4782969/4194304	1.140348673
$D_\sharp^{(A)}, E_\flat^{(A)}$	x^1y^2	$2^{-14}3^9$	19683/16384	1.201354981
$E^{(A)}$	x^3y^1	$2^{13}3^{-8}$	8192/6561	1.248590154
$F^{(A)}$	x^3y^2	2^23^{-1}	4/3	1.333333333
$F_\sharp^{(A)}$	x^4y^2	$2^{10}3^{-6}$	1024/729	1,404663923
$G_\flat^{(A)}$	x^3y^3	$2^{-9}3^6$	729/512	1.423828125
$G^{(A)}$	x^4y^3	$2^{-1}3^1$	3/2	1.5
$G_\sharp^{(A)}, A_\flat^{(A)}$	x^4y^4	$2^{-12}3^8$	6561/4096	1.601806641
$A^{(A)}$	x^6y^3	$2^{15}3^{-9}$	32768/19683	1.664786873
	x^7y^3	$2^{23}3^{-14}$	8388608/4782969	1.753849545
$A_\sharp^{(A)}, B_\flat^{(A)}$	x^6y^4	2^43^{-2}	16/9	1.777777777
	x^5y^5	$2^{-15}3^{10}$	59049/32768	1.802032471
$B^{(A)}$	x^7y^4	$2^{12}3^{-7}$	4096/2187	1.872885232
$C^{(A)}$	x^7y^5	2^13^0	2/1	2.0

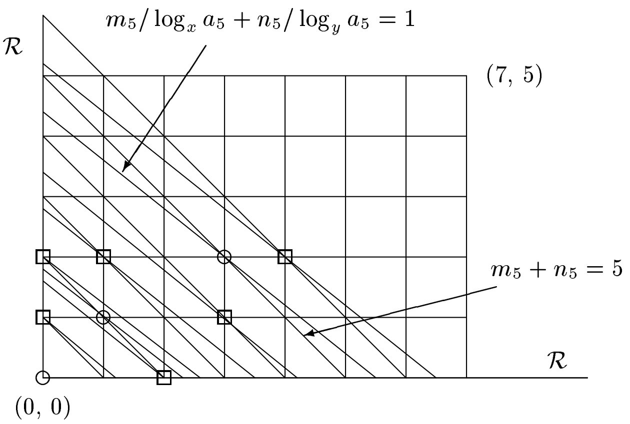

Fig. 2. JI $(C, \ldots, F_\sharp)$ in the map θ_*.

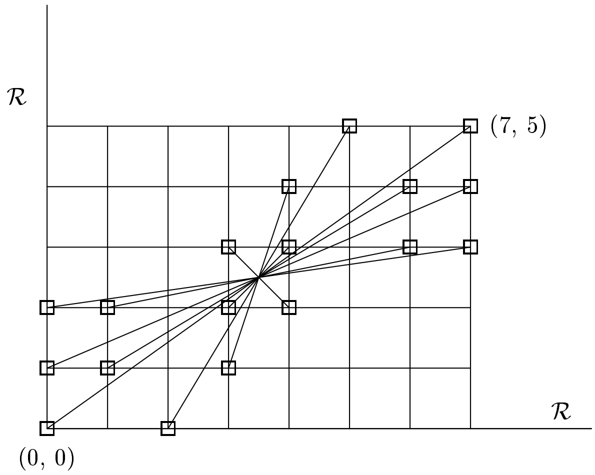

Fig. 3. Symmetry of the JI Approximation $\mathcal{A}$ in the isomorphism θ.

The fact that the JI Approximation $\mathcal{A}$ is a generalization of Equal Temperament, we prove in the following theorem which states that the image of Equal Temperament is a projection of the JI Approximation to a line in the map θ_*.

Theorem 4. *Let p, q be two lines in the complex plane, such that $(0,0),(7,5) \in p$; $(12,0),(7,5) \in q$, respectively. Let $\pi : \mathcal{C} \to p$ be the projection of the plane $\mathcal{C}$ into the line p along the line q. Let $\mathcal{S}$ be an arbitrary 12-degree 2-interval (2/1, 3/2)-system. Then $W = \theta_*^{-1}(\pi(\eta(\theta(\mathcal{S}))))$.*

Proof. Cf. Fig. 4. Without loss of generality, we prove the assertion for $\mathcal{S} = \mathcal{A}$.
Denote by $w_0 = (0,0), w_1 = (\frac{7}{12}, \frac{5}{12}), w_2 = 2 \cdot (\frac{7}{12}, \frac{5}{12}), \ldots, w_{12} = 12 \cdot (\frac{7}{12}, \frac{5}{12})$.
We have:

$$\begin{aligned}
w_0 &= \pi(\eta(\theta(C^{(A)}))), & \theta_*^{-1}(w_0) &= x^0 y^0 &&= 1,\\
w_1 &= \pi(\eta(\theta(C_\sharp^{(A)}))), & \theta_*^{-1}(w_1) &= x^{\frac{7}{12}} y^{\frac{5}{12}} &&= \sqrt[12]{2},\\
w_2 &= \pi(\eta(\theta(D^{(A)}))), & \theta_*^{-1}(w_2) &= x^{2\cdot\frac{7}{12}} y^{2\cdot\frac{5}{12}} &&= (\sqrt[12]{2})^2,\\
& & \ldots & & &\\
w_{12} &= \pi(\eta(\theta(C'^{(A)}))), & \theta_*^{-1}(w_{12}) &= x^{12\cdot\frac{7}{12}} y^{12\cdot\frac{5}{12}} &&= (\sqrt[12]{2})^{12}.
\end{aligned}$$

So, $W = \theta_*^{-1}(\pi(\eta(\theta(\mathcal{S}))))$. □

4. Comma 32 805/32 768

Now we show that the JI Approximation $\mathcal{A}$ is closer to JI than Equal Temperament. As a unit q for imagination of measure values, we take

$$q = \log_2(G^{(W)}/G^{(JI)}) = \log_2 3 - (19/12) \approx 0.001629167$$

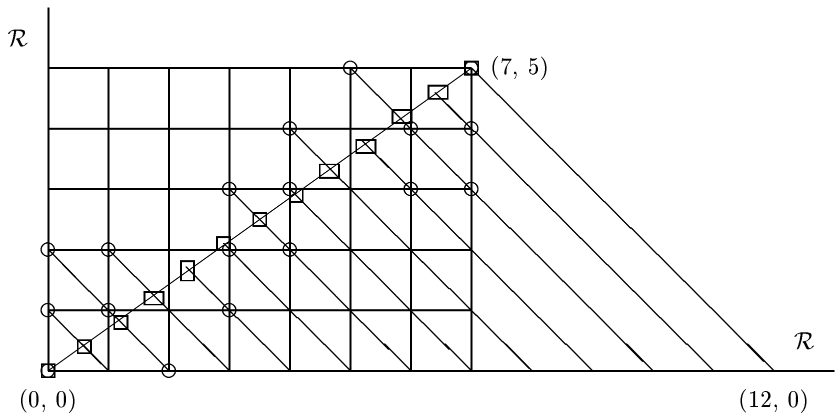

Fig. 4. Image of Equal Temperament in the map θ_*.

Table 4. The JI Approximation $\mathcal{A}$ versus Equal Temperament.

	$\rho_q(\cdot^{(JI)}, \cdot^{(A)})$	$\rho_q(\cdot^{(JI)}, \cdot^{(W)})$
C	0	0
$C_\sharp = D_\flat$	0.999342947	6.000641318
D 10/9	0.999342947	9.000633554
D 9/8	0	1.999995685
D 8/7	1.945816399	15.94578457
$D_\sharp, E_\flat$	0.999342947	7.000637399
E	0.999342947	7.000637399
F	0	1
$F_\sharp$	0.999342947	5.000641211

for the metric ρ_q, cf. Lemma 1 and also [10], [11]. Clearly,

$$\rho_q(G^{(W)}/G^{(JI)}) = 1.$$

Note that the equal tempered fifth is the best approximated (to JI) music interval in Equal Temperament (not mentioning trivially the unison and octave).

Theorem 5. *For each (qualitative music) interval $C, C_\sharp, \ldots, B, C'$, the JI Approximation $\mathcal{A}$ value is closer to JI value than the Equal Temperament value in the metric ρ_q with $q = \log_2 3 - (19/12)$.*

Proof. Use Theorem 2, cf. Table 4. □

The following lemma shows that the JI Approximation $\mathcal{A}$ is characterized by a "comma" κ, where

$$\kappa = \mathbf{5 \cdot 3^8 \cdot 2^{-15} = 32\ 805/32\ 768 \approx 1.00112915}.$$

We immediately obtain:

Theorem 6. *Let* $\kappa = 32\ 805/32\ 768$. *Let* $q = \log_2 3 - (19/12)$. *Then*

$$\begin{aligned} \log_2 \sqrt[q]{\kappa} &= \rho_q(C_\sharp^{(A)}, C_\sharp^{(JI)}) = \rho_q(D_\sharp^{(A)}, D_\sharp^{(JI)}) = \rho_q(E^{(A)}, E^{(JI)}) = \rho_q(F_\sharp^{(A)}, F_\sharp^{(JI)}) \\ &= \rho_q(D^{(A)}(10/9), D^{(JI)}(10/9)) = \rho_q(B^{(A)}(9/5), B^{(JI)}(9/5)) \\ &= \rho_q(G_\flat^{(A)}, G_\flat^{(JI)}) = \rho_q(A_\flat^{(A)}, A_\flat^{(JI)}) = \rho_q(A^{(A)}, A^{(JI)})\rho_q(B^{(A)}, B^{(JI)}) \\ &< 0.999345389 < 1 = \rho_q(G^{(W)}, G^{(JI)}). \end{aligned}$$

We collect the results of the paper into the following theorem.

Theorem 7. *Let* $x = 2^8/3^5, y = 3^7/2^{11}$. *Let* $\kappa = 32\ 805/32\ 768$. *Then the JI Approximation*

$$\begin{gathered} \mathcal{A} = \{\ (x^4y^3)^k(x^7y^5)^{-h};\ (k,h) = \\ [-14,-9], \\ (-10,-6),(-9,-6),(-8,-5),(-7,-5),(-6,-4), \\ \{-2,-2\},\{-1,-1\},\{0,-1\},\ \ \{0,0\},\{1,0\},\{2,1\}, \\ (6,3),(7,4),(8,4),(9,5),(10,5), \\ [14,8]\ \}, \end{gathered}$$

where

- $\{\cdot,\cdot\}$ *denotes approximated intervals based on numbers 2 and 3*

  $$(B_\flat^{(A)}(16/9), F^{(A)}, C'^{(A)};\ C^{(A)}, G^{(A)}, D^{(A)}(9/8),$$

  respectively) and the JI Approximation $\mathcal{A}$ *coincides with the JI value;*

- $(\cdot,\cdot)$ *denotes approximated intervals based on numbers 2, 3, and 5*

  $$(D^{(A)}(10/9), A^{(A)}, E^{(A)}, B^{(A)}, F_\sharp^{(A)};\ G_\flat^{(A)}, C_\sharp^{(A)}, A_\flat^{(A)}, E_\flat^{(A)}, B_\flat^{(A)}(9/5),$$

  respectively) and the intervals are κ*-approximated;*

- $[\cdot,\cdot]$ *denotes the JI Approximated intervals based on numbers 2 and 7*

  $$(B_\flat^{(A)}(7/4);\ D^{(A)}(8/7),$$

  respectively).

Proof. The vector (k, h) is a solution of the following system of equations:

$$4k + (-7)h = \alpha, 3k + (-5)h = \beta,$$

where α, β are exponents of x, y, respectively, cf. Definition 2 (e.g. $(3, 2)$ for F). Observe that

$$\det \begin{pmatrix} 4 & -7 \\ 3 & -5 \end{pmatrix} = 1,$$

so $k, h \in \mathcal{Z}$ and $(k, h) = (-5\alpha + 7\beta, -3\alpha + 4\beta)$. Rearrange now the solutions (k, h) as above. □

Remark 2. A tone system based on the (-14)th, (-8)th, 0th, 1st, and 9th fifths is called the *Petzval tone system of the Second class*, cf. [12]. We see, that the JI Approximation $\mathcal{A}$ is the many-valued system of this type.

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