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On two algorithms in music acoustics

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Abstract

There is shown that the two basic tuning algorithms of key instruments in music (the fifths and middle-tone tunings) can be described via a unifying mathematical theory.

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1 The fifths and middle-tone tunings

When tuning key instruments in music, two basic algorithms are known. The first one is based on the perfect fifths, the second one on the major thirds. In their pure form, these algorithms lead to two different classical tone systems - to the Pythagorean (Pythagoras, 582 – 492 B. C.) and Praetorius Tunings (Michael Praetorius [Schultze], 1571 – 1621). Each other 12-tone system used in the European music can be obtained by a modification (*temperament*) of the algorithm of Pythagorean or the Praetorius Tunings (or the mixture of both of them). The excellent example of that is the widely spread Equal Temperament which modifies the Pythagorean Tuning algorithm such that the interval of every 12 (tempered) fifths in this tuning gives the interval of the 7 perfect octaves.

For the sake of tuning, it is reasonable to identify the tone pitch with its relative frequency to the frequency of a fundamental, fixed tone (conventionally, such a tone is usually taken $a^1 = 440$ Hz in the experience; we take $c = 1$ for simplicity). So, in fact, we deal with relative frequencies of music intervals.

Describe the algorithm of Praetorius Tuning. Starting with a frequency q_1 and both divide and multiply q_1 by $5/4$, we obtain the relative frequencies q_2, q_3, respectively. The next step, the frequency q_4, we obtain by the geometrical mean $\sqrt{q_1 q_2}$, etc., see Figure 1. If we replace the sequence $q_1, q_2, \cdots, q_{12}$ by the obvious musical notation (we will use small letters for Praetorius Tuning), we obtain the sequence:

$$e, c, g_\sharp, d, b_\flat, f_\sharp, a, f, c_\sharp, g, e_\flat, b.$$

All other frequencies of Praetorius Tuning we obtain via the so called *octave equivalency*, i.e. multiplying the described 12 values by 2^i, i is an integer. The relative values of Praetorius Tuning

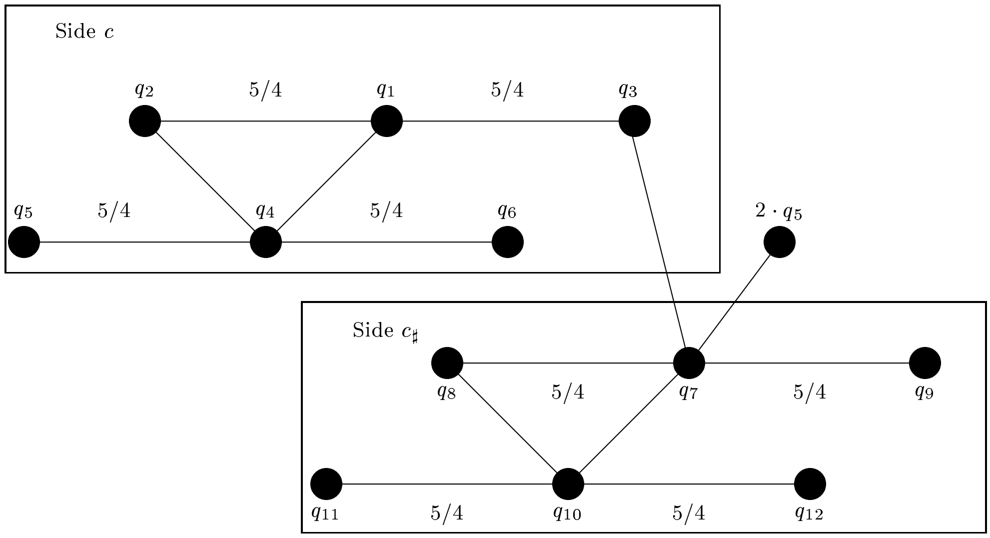

Figure 1: Algorithm of Praetorius Tuning is based on the major thirds

within the interval $[1, 2], c = 1$, are in the Table 1. The set of pipes of an organ is divided usually into two parts (often, two boxes) – "Side c" and "Side $c_\sharp$". The reason of this arrangement is the algorithm of Praetorius tuning which prevailed in the 17th century, the golden age of organ.

Describe the algorithm of Pythagorean Tuning. Start with a frequency $q_1 \in [1, 2)$. The n-th step, the frequency q_n is evaluated as follows:

$$q_n = q_{n-1}\frac{3}{2} \text{ if } q_{n-1} \cdot \frac{3}{2} < 2$$

or

$$q_n = q_{n-1}\frac{3}{2} \cdot \frac{1}{2} \text{ if } q_{n-1} \cdot \frac{3}{2} \geq 2.$$

If $q_7 = 1$, then replacing $p_1, p_2, \cdots, p_{17}$ by the obvious music notation (we will use the capital letters for Pythagorean Tuning), we obtain the sequence

$$G_\flat, D_\flat, A_\flat, E_\flat, B_\flat, F, C, G, D, A, E, B, F_\sharp, C_\sharp, G_\sharp, D_\sharp, A_\sharp.$$

Again, we enlarge these frequencies to the whole halfline $(0, \infty)$ (in fact, 16 - 20 000 Hz) via the octave isomorphism. The values within the interval $[1, 2), C = 1$, are in the Table 2.

Pythagorean Tuning is usually mentioned as to be 17-valued (also: 5, 7, 12 - valued, see Figure 2). We reduce the 17-valued Pythagorean Tuning to 12-values, (and $Y_1 = 256/243, Y_2 = 2187/2048$ in Definition 4), the values are marked by $*$ in Table 2. An other approach to Pythagorean Tuning is presented in [3].

2 Semitones

Pythagorean Tuning has two semitones, expressed via the unique rationals:

$$Y_1 = 256/243, Y_2 = 2187/2048$$

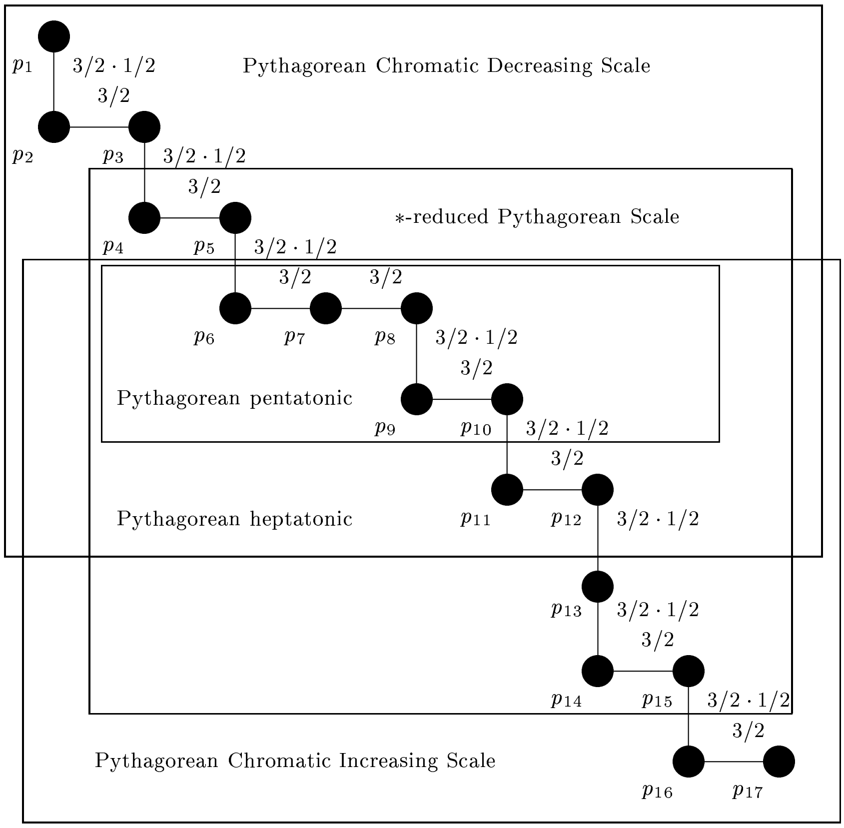

Figure 2: Algorithm of Pythagorean Tuning is based on the perfect fifths

(the minor Pythagorean semitone, *diesis* and the major Pythagorean semitone, *apotome*, and contains only harmonic music intervals (intervals expressed by rationals), see Table 2 and [1].

Praetorius Tuning (also known as the Middle-tone Tuning), [4], contains both harmonic (octaves, major thirds, minor sixths) and inharmonic music intervals, see Table 1.

Excluding octaves, Equal Temperament contains only inharmonic intervals and is represented by the geometrical progression $\langle \sqrt[12]{2^i} \rangle$ with the quotient $\sqrt[12]{2}$, the *equal tempered semitone*.

About semitones (and other constructing intervals) of the the diatonic scales in general, see [2].

In the present paper there is shown the existence and uniqueness of semitones of Praetorius Tuning and an isomorphism between Pythagorean and Praetorius Tunings.

Theorem 1 *The semitones of Praetorius Tuning are given by numbers:*

$$\sqrt[4]{78125}/16, 8/\sqrt[4]{3125}.$$

More precisely (see Definitions 1 and 3, Proof of Theorem 2):

Theorem 2 *According to the symmetry, there exist a unique pair of algebraic numbers* (X_1, X_2)

$X_2^0 X_1^0$	$2^0 5^0$	1	1.0	0.0000	c
$X_2^0 X_1^1$	$2^{-4} 5^{7/4}$	$\sqrt[4]{78125}/16$	1.044907	76.0490	$c_\sharp$
$X_2^1 X_1^1$	$2^{-1} 5^{1/2}$	$\sqrt[2]{5}/2$	1.118034	193.1569	d
$X_2^2 X_1^1$	$2^2 5^{-3/4}$	$4/\sqrt[4]{125}$	1.196279	310.2647	$e_\flat$
$X_2^2 X_1^2$	$2^{-2} 5^1$	$5/4$	1.25	386.3137	e
$X_2^3 X_1^2$	$2^1 5^{-1/4}$	$2/\sqrt[4]{5}$	1.337481	503.4219	f
$X_2^3 X_1^3$	$2^{-3} 5^{1/2}$	$\sqrt[2]{125}/8$	1.397542	579.4706	$f_\sharp$
$X_2^4 X_1^3$	$5^{1/4}$	$\sqrt[4]{5}$	1.495349	696.5784	g
$X_2^4 X_1^4$	$2^{-4} 5^2$	$25/16$	1.5625	772.6274	$g_\sharp$
$X_2^5 X_1^4$	$2^{-1} 5^{3/4}$	$\sqrt[4]{125}/4$	1.671851	889.7353	a
$X_2^6 X_1^4$	$2^2 5^{-1/2}$	$4/\sqrt[2]{5}$	1.788854	1006.8430	$b_\flat$
$X_2^6 X_1^5$	$2^{-2} 5^{5/4}$	$\sqrt[4]{3125}/4$	1.869186	1082.8920	b
$X_2^7 X_1^5$	2^1	2	2.0	1200.0000	c'

Table 1: Praetorius Tuning

$Y_1^0 Y_2^0$	$2^0 3^0$	1	1.0	0.0000	C	$*$
$Y_1^1 Y_2^0$	$2^8 3^{-5}$	$256/243$	1.053497942	90.2250	$D_\flat$	
$Y_1^0 Y_2^1$	$2^{-11} 3^7$	$2187/2048$	1.067871094	113.6850	$C_\sharp$	$*$
$Y_1^1 Y_2^1$	$2^{-3} 3^2$	$9/8$	1.125	203.9100	D	$*$
$Y_1^2 Y_2^1$	$2^5 3^{-3}$	$32/27$	1.185185185	294.1350	$E_\flat$	$*$
$Y_1^1 Y_2^2$	$2^{-14} 3^9$	$19683/16384$	1.201354981	317.5950	$D_\sharp$	
$Y_1^2 Y_2^2$	$2^{-6} 3^4$	$81/64$	1.265625	407.8200	E	$*$
$Y_1^3 Y_2^2$	$2^2 3^{-1}$	$4/3$	1.333333333	498.04500	F	$*$
$Y_1^4 Y_2^2$	$2^{10} 3^{-6}$	$1024/729$	1.404663923	588.2700	$G_\flat$	
$Y_1^3 Y_2^3$	$2^{-9} 3^6$	$729/512$	1.423828125	611.7300	$F_\sharp$	$*$
$Y_1^4 Y_2^3$	$2^{-1} 3^1$	$3/2$	1.5	701.9550	G	$*$
$Y_1^5 Y_2^3$	$2^7 3^{-4}$	$128/81$	1.580246914	792.1800	$A_\flat$	
$Y_1^4 Y_2^4$	$2^{-12} 3^8$	$6561/4096$	1.601806641	815.9039	$G_\sharp$	$*$
$Y_1^5 Y_2^4$	$2^{-4} 3^3$	$27/16$	1.6875	905.8650	A	$*$
$Y_1^6 Y_2^4$	$2^4 3^{-2}$	$16/9$	1.777777777	996.0900	$B_\flat$	$*$
$Y_1^5 Y_2^5$	$2^{-15} 3^{10}$	$59049/32768$	1.802032473	1019.5500	$A_\sharp$	
$Y_1^6 Y_2^5$	$2^{-7} 3^5$	$243/128$	1.898437528	1109.7750	B	$*$
$Y_1^7 Y_2^5$	2^1	2	2.0	1200.0000	C'	$*$

Table 2: The $*$-reduced Pythagorean Tuning

for $\mathcal{M}$-generalized geometrical progressions which yields Praetorius Tuning, where

$$X_1 = \frac{5^{7/4}}{2^4} = \sqrt[4]{78125}/16 \approx 1.044906726, X_2 = \frac{2^3}{5^{5/4}} = 8/\sqrt[4]{3125} = 1.069984488.$$

Combine these results with the algorithms above and consequently collect Table 1 (in the fifth column, there are values in cents, i.e. in the isomorphism $\Gamma_i \mapsto 1200 \cdot \log_2 \Gamma_i$; in the sixth column, there is a musical denotation).

The music interval $e_\flat/g_\sharp$ is the famous so called *wolf fifth* in Praetorius Tuning.

The following theorem is very interesting since: (1) Pythagorean and Praetorius Tunings are obtained by two very different algorithms; (2) it leads to many consequences in music theory. The assertion follows when comparing Table 1 and Table 2.

Theorem 3 *Praetorius and $*$-reduced Pythagorean Tunings (given by Definitions 3 and 4) are isomorphic (in the sense of Definition 1).*

3 Definitions and Proofs

Denote by $\mathcal{N} = \{0, 1, 2, \ldots\}$. If we denote by $\mathcal{L} = ((0, \infty), \cdot, 1, \leq)$ the usual multiplicative group on reals with the usual order, then the $\mathcal{L}$-length b/a of the interval $(a, b), 0 < a \leq b < \infty$ we call *the music interval*.

We will use the following conventional notation:

$$X = (X_1, X_2, \cdots, X_n) \in \mathcal{L}^n, \nu_{i,\cdot} = (\nu_{i,1}, \nu_{i,2}, \ldots, \nu_{i,n}) \in \mathcal{N}^n, |\nu_{i,\cdot}| = \nu_{i,1} + \nu_{i,2} + \ldots + \nu_{i,n},$$

$$\nu_{i,\cdot} \leq \nu_{i+1,\cdot} \Leftrightarrow \nu_{i,k} \leq \nu_{i+1,k} \ (k = 1, 2, \ldots, n), X^{\nu_{i,\cdot}} = X_1^{\nu_{i,1}} X_2^{\nu_{i,2}} \ldots X_n^{\nu_{i,n}} \quad (i, n \in \mathcal{N}).$$

Definition 1 For $n \in \mathcal{N}$, we say that a sequence $\langle \Gamma_i \rangle$ is an *n-generalized geometrical progression* if there exist $X \in \mathcal{L}^n$ and $\nu_{i,j} \in \mathcal{N}$ $(i \in \mathcal{N}, j = 1, 2, \ldots, n)$ such that

$$\Gamma_i = X^{\nu_{i,\cdot}}, \nu_{0,\cdot} \leq \nu_{1,\cdot} \leq \ldots \leq \nu_{i,\cdot} \leq \ldots, |\nu_{i,\cdot}| = i.$$

We say that two n-generalized geometrical progressions $\langle \Gamma_i \rangle, \langle \Delta_i \rangle$ are *isomorphic* if there exist $\nu_{i,\cdot} \in \mathcal{N}, X, Y \in \mathcal{L}^n$, such that

$$\Gamma_i = X^{\nu_{i,\cdot}}, \Delta_i = Y^{\nu_{i,\cdot}}.$$

In this paper, we reduce the general situation to the case $n = 2$.

Definition 2 Let $k = 0, 1, 2, ..., 11, 12$. We say that a matrix $(\nu_{i,j})_{12,k}^{1,2} \in \mathcal{N}^2 \times \mathcal{N}^2$ is a *$(12, k)$-matrix*, if

$$0 \leq \nu_{k,\cdot} \leq \nu_{12,\cdot}, \quad |\nu_{i,\cdot}| = i, i = 12, k.$$

The following definition is based on the algorithm of Praetorius Tuning.

Definition 3 We say that a 2-generalized geometrical progression $\langle \Gamma_i \rangle$ is *$\mathcal{M}$-generated by a (12, 4)-matrix* if for some $X_1, X_2 > 0$,

$$\nu_{0,\cdot} = 0, \nu_{1,\cdot} = \frac{7}{4}\nu_{4,\cdot} - \frac{\nu_{12,\cdot}}{2}, \nu_{2,\cdot} = \frac{\nu_{4,\cdot}}{2}, \nu_{3,\cdot} = \frac{\nu_{12,\cdot}}{2} - \frac{3}{4}\nu_{4,\cdot}, \nu_{5,\cdot} = \frac{\nu_{12,\cdot}}{2} - \frac{\nu_{4,\cdot}}{4},$$

$$\nu_{6,\cdot} = \frac{3}{2}\nu_{4,\cdot}, \nu_{7,\cdot} = \frac{\nu_{4,\cdot}}{4} + \frac{\nu_{12,\cdot}}{2}, \nu_{8,\cdot} = 2 \cdot \nu_{4,\cdot}, \nu_{9,\cdot} = \frac{3}{4}\nu_{4,\cdot} + \frac{\nu_{12,\cdot}}{2},$$

$$\nu_{10,\cdot} = \nu_{12,\cdot} - \frac{\nu_{4,\cdot}}{2}, \nu_{11,\cdot} = \frac{5}{4}\nu_{4,\cdot} + \frac{\nu_{12,\cdot}}{2}$$

and for $i \geq 12$, there exists $p \in \mathcal{N}, 0 \leq p < 12$, and $q \in \mathcal{N}$, such that $\nu_{i,\cdot} = q\nu_{12,\cdot} + \nu_{p,\cdot}$.

The following definition is based on the algorithm of Pythagorean Tuning (it defines the $*$-reduced Pythagorean Scale when Y_1, Y_2 are diesis and apotome, respectively).

Definition 4 We say that a 2-generalized geometrical progression $\langle \Gamma_i \rangle$ is *$*$-generated by a (12, 7)-matrix* if for some $Y_1, Y_2 > 0$ the first 12 of Γ_i is evaluated as follows (reordering increasingly)

$$\{(Y_1^{\nu_{7,1}} Y_2^{\nu_{7,2}})^k (\text{mod } Y_1^{\nu_{12,1}} Y_2^{\nu_{12,2}})\}_{k=3,\ldots,14},$$

and for $i \geq 12$, there exists $p \in \mathcal{N}, 0 \leq p < 12$, and $q \in \mathcal{N}$, such that $\nu_{i,\cdot} = q\nu_{12,\cdot} + \nu_{p,\cdot}$, cf. [1].

Proof of Theorem 2. Suppose that there exist $X_1, X_2 > 0$ in Definition 3. Then it is easy to see that Definition 3 implies Praetorius Tuning if both X_1, X_2 are algebraic.

Prove that there exist unique algebraic $X_1, X_2 > 0$ satisfying Definition 1. To show this, consider the following equation system

$$\begin{aligned} X_1^{\nu_{12,1}} X_2^{\nu_{12,2}} \omega &= 2 \\ X_1^{\nu_{7,1}} X_2^{\nu_{7,2}} \omega &= \sqrt[4]{5} \\ X_1^{\nu_{4,1}} X_2^{\nu_{4,2}} \omega &= \tfrac{5}{4} \end{aligned}$$

where ω is an arbitrary real number (a parameter, a shift when tuning) and values $2, \sqrt[4]{5}, \frac{5}{4}$ are given by Praetorius Tuning, [4], (the octave, the minor third and the Pretorius fifth). We have:

$$\begin{aligned} \nu_{12,1}\log X_1 + \nu_{12,2}\log X_2 + \log\omega &= \log 2 \\ \nu_{7,1}\log X_1 + \nu_{7,2}\log X_2 + \log\omega &= \log\sqrt[4]{5} \\ \nu_{4,1}\log X_1 + \nu_{4,2}\log X_2 + \log\omega &= \log\tfrac{5}{4}. \end{aligned}$$

Consider $\nu_{12,1}, \nu_{4,1} \in \mathcal{N}$, such that

$$\begin{gathered} 0 \le \nu_{1,1} \le \frac{7}{4}\nu_{4,1} - \frac{\nu_{12,1}}{2} \\ \le \frac{\nu_{4,1}}{2} \le \frac{\nu_{12,1}}{2} - \frac{3}{4}\nu_{4,1} \\ \le \nu_{4,1} \le \frac{\nu_{12,1}}{2} - \frac{\nu_{4,1}}{2} \le \frac{3}{2}\nu_{4,1} \le \frac{\nu_{4,1}}{4} + \frac{\nu_{12,1}}{2} \\ \le 2\cdot\nu_{4,1} \le \frac{3}{4}\nu_{4,1} + \frac{\nu_{12,1}}{2} \le \frac{-\nu_{4,1}}{2} + \nu_{12,1}, \\ \le \frac{5}{4}\nu_{4,1} + \frac{\nu_{12,1}}{2} \le \nu_{12,1} \le 12. \end{gathered}$$

The condition $\nu_{2,1} = \nu_{4,1}/2 \le 4$ implies possibilities $\nu_{4,1} = 2, 4$ ($\nu_{4,1} \ne 0$ since otherwise $\nu_{1,1} < 0$). To $\nu_{4,1} = 2$ we have possibilities $\nu_{12,1} = 3, 5, 7$ (the last two are symmetric). To $\nu_{4,1} = 4$ we have possibilities $\nu_{12,1} = 2, 6, 8, 10$. From these cases only the unique pair $(\nu_{12,1}, \nu_{4,1}) = (5, 2)$ (symmetrically $(7, 2)$) is such that it may $\mathcal{M}$-generate a 2-generated geometrical progression.

Solve the equation system above for $\nu_{12,1} = 5, \nu_{4,1} = 2$. By Definition 1 and Definition 3, the determinant of this equation system is

$$\begin{vmatrix} \nu_{12,1} & 12-\nu_{12,1} & 1 \\ \nu_{7,1} & 7-\nu_{7,1} & 1 \\ \nu_{4,1} & 4-\nu_{4,1} & 1 \end{vmatrix} = \begin{vmatrix} \nu_{12,1} & 12 & 1 \\ \nu_{7,1} & 7 & 1 \\ \nu_{4,1} & 4 & 1 \end{vmatrix} =$$

$$= 3\nu_{12,1} - 8\nu_{7,1} + 5\nu_{4,1} = \nu_{12,1} + 5\nu_{4,1} - 8(\nu_{4,1}/4 + \nu_{12,1}/2) = 3\nu_{4,1} - \nu_{12,1} = 1.$$

Consequently,

$$(X_1, X_2, \omega) = \left(\frac{5^{7/4}}{2^4}, \frac{2^3}{5^{5/4}}, 1\right) =$$

$$\left(\sqrt[4]{78125}/16, 8/\sqrt[4]{3125}, 1\right) \approx (1.044906726, 1.069984488, 1.0).$$

We obtained also, that $\omega = 1$ (there is no shift of the fundamental tone when tuning). □

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