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ON INTEGRATION IN COMPLETE VECTOR LATTICES

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ABSTRACT. For the Archimedean vector lattice $\mathbf{X}$, the complete vector lattice $\mathbf{Y}$ and the positive cone $\mathbb{L}$ of the vector lattice of all linear regular operators $L\colon \mathbf{X} \to \mathbf{Y}$, a Riemann-type construction of integral for $\mathbb{L}$-valued measures is discussed. Moreover, if $\mathbf{Y}$ is almost regular, a convergence theorem is proved.

INTRODUCTION

Riemann-type concepts, cf. [14], of the integral have their new Renaissance in the vector integration theory. E.g., let $\mathbf{X}$ be a linear metric space but not locally convex, and $\mathbf{m}(\cdot)\mathbf{x} = \mathbf{x}\lambda(\cdot), \mathbf{x} \in \mathbf{X}$, where λ is a Lebesgue measure on the real line. It is clear that (because of the convergence of simple functions) a Lebesgue integral cannot be principally defined for the measure $\mathbf{m}$, although Riemann integrals can be defined very well, cf. [6].

Denote by $\mathbb{R}, \mathbb{N}$ the real line and the set of all naturals, respectively. Let $\Delta : [a,b] \to (0,\infty),\ a,b \in \mathbb{R}, a \leq b$, be a real function. Let $\mathcal{A}(\Delta)$ be the family of all partitions $\mathcal{D} = \{(E_1,t_1),\dots,(E_J,t_J)\}$, such that $E_j \subset (t_j - \Delta(t_j), t_j + \Delta(t_j))$, where the sets $E_j, j = 1,2,\dots,J$, are non-overlapping compact subintervals of the interval $[a,b]$ covering $[a,b]$, and $t_j \in E_j, j = 1,2,\dots,J$, are chosen points. Let λ be the Lebesgue measure. A function $f : [a,b] \to \mathbb{R}$ is *Kurzweil integrable* if there exists a constant $c \in \mathbb{R}$ and for every $\varepsilon > 0$ a real function $\Delta : [a,b] \to (0,\infty)$, such that for every $\mathcal{D} \in \mathcal{A}(\Delta)$ the inequality $|c - \sum_{j=1}^{J} f(t_j)\lambda(E_j)| < \varepsilon$ holds, cf. [7], Definitions 3.22, 3.24. It is known that a function is Kurzweil integrable on $[a,b]$ if and only if it is Perron integrable on $[a,b]$ and a non-negative function is Kurzweil integrable on $[a,b]$ if and only if it is Lebesgue integrable on $[a,b],\ a,b \in \mathbb{R}, a \leq b$, cf. [5].

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S. I. Ahmed and W. F. Pfeffer, [1], and B. Riečan, [8], defined a Kurzweil integral for real functions and for a non-negative Borel measure in locally compact topological spaces. Further B. Riečan, [9], [10]; B. Riečan and M. Vrábelová, [11], [12], developed a generalized Kurzweil integration to ordered spaces with some properties.

There are few constructions of integral with respect to operator valued measures without order structures, cf. [3]. These constructions are based on the notion of semivariation (or variation) of a measure. However, this quantity cannot be practically computed without any additional property, e.g. again considering some order structure. Our idea is to choose suitable ideals of spaces in which the semivariation coincides with an operator norm. The choice of these ideals may depend on the function which we integrate. An integral is viewed in the present paper as a linear map from a lattice of functions into another lattice. For the Archimedean vector lattice $\mathbf{X}$ and the complete vector lattice $\mathbf{Y}$, we construct a Riemann integral with respect to an $\mathbb{L}$-valued measure, where $\mathbb{L}$ is the positive cone of the vector lattice of all linear regular operators $L : \mathbf{X} \to \mathbf{Y}$. Moreover, if $\mathbf{Y}$ is almost regular, a convergence theorem is proved.

1. Preliminaries

For notation and terminology concerning Riesz spaces we refer to monographs [2], [4], and [13].

Let $\mathbf{X}$ be an Archimedean vector lattice and $\mathbf{X}^+ = \{\mathbf{x} \in \mathbf{X}; \mathbf{x} = \mathbf{x} \vee 0\}$. A sequence $(\mathbf{x}_i)_{i\in\mathbb{N}}$ is said to be (r)-*convergent* to $\mathbf{x} \in \mathbf{X}$ (we write $\mathbf{x} = (r)\text{-}\lim_{i\in\mathbb{N}} \mathbf{x}_i$), if there exists $\rho \in \mathbf{X}^+$ (called *the regularizator*), such that $\forall\varepsilon > 0, \exists i_0 \in \mathbb{N}, \forall i \geq i_0 : |\mathbf{x} - \mathbf{x}_i| \leq \varepsilon\rho$.

Each linear solid subspace of a vector lattice is called *an ideal.* If $\rho \in \mathbf{X}^+$, then the smallest ideal containing ρ is called *an* (ρ)*-ideal* in $\mathbf{X}$ (denoted by $\mathbf{X}(\rho)$). A norm $\|\cdot\|$ defined on a vector lattice $\mathbf{X}$ is *monotone* if $|\mathbf{x}_1| \leq |\mathbf{x}_2| \implies \|\mathbf{x}_1\| \leq \|\mathbf{x}_2\|$. It is easy to see, that $\mathbf{X}(\rho) = \{\mathbf{x} \in \mathbf{X}; \exists\lambda, 0 \leq \lambda < \infty, |\mathbf{x}| \leq \lambda\rho\}$ and the Minkowski functional $\|\cdot\|_\rho$ of $[-\rho, \rho], \rho > 0$, is a monotone norm in $\mathbf{X}(\rho)$. In normed lattices the convergence with a regularizator implies the norm convergence, cf. [2], p. 377. Evidently, if $\rho \in \mathbf{X}, \rho > 0, \mathbf{x} \in \mathbf{X}(\rho)$, then $|\mathbf{x}| \leq \|\mathbf{x}\|_\rho \cdot \rho$.

A vector lattice $\mathbf{Y}$ is said to be *complete* if every set bounded from above has a supremum. An ideal in a complete vector lattice is a complete vector lattice, too. A complete vector lattice $\mathbf{Y}$ is said to be *almost regular* if the (r)-convergence on $\mathbf{Y}$ is equivalent to the (o)-convergence on $\mathbf{Y}$ (defined as usually). Note, that B. Z. Vulikh use the adjective "almost regular" only for complete vector lattices of countable type, cf. [13], VI, §4.

Example 1.1. If $\mathbf{Y}$ is a complete finite dimensional vector lattice with unit, then $\mathbf{Y}$ is almost regular, cf. [13], Th. VI.4.2.

Example 1.2. Let $\mathbf{Y}$ be the complete vector lattice of all sequences of reals, such that $(r_n)_{n\in\mathbb{N}} \in \mathbf{Y}$ iff $\exists i_1,\dots,i_k \in \mathbb{N} : r_{i_1},\dots,r_{i_k} \neq 0$ and $r_i = 0,\ i \in \mathbb{N}\setminus\{i_1,\dots,i_k\}, k \in \mathbb{N}$, (the positive cone of $\mathbf{Y}$ is defined as usually). Then $\mathbf{Y}$ is almost regular and is not regular, cf. [13], VI.§5.

Let T be a non-trivial Hausdorff topological space which is to serve as a basis space. We denote by $cl(E)$ the closure of a set $E \subset T$. For $E \subset T, \rho \in \mathbf{X}, \rho > 0$, define the following seminorm $\|.\|_{E,\rho} : \mathbf{X}(\rho)^T \to [0,\infty]$, where $\|\mathbf{f}\|_{E,\rho} = \sup_{t\in E} \|\mathbf{f}(t)\|_\rho, \mathbf{f} : T \to \mathbf{X}(\rho)$.

Let $\mathbf{X}, \mathbf{Y}$ be two real Archimedean vector lattices. Let $L(\mathbf{X},\mathbf{Y})$ be a space of all linear regular operators $L : \mathbf{X} \to \mathbf{Y}$, cf. [13], Definition VIII.1.2. Let $\mathbb{L}$ be *the positive cone* of $L(\mathbf{X},\mathbf{Y})$, i.e. $L \in \mathbb{L}$ if and only if for every $\mathbf{x} \in \mathbf{X}^+$ there is $L\mathbf{x} \in \mathbf{Y}^+ = \{\mathbf{y} \in \mathbf{Y}; \mathbf{y} = \mathbf{y}\vee 0\}$. Every additive and positively homogeneous operator $L_0 : \mathbf{X}^+ \to \mathbf{Y}^+$, has a unique extension to a linear operator $L : \mathbf{X} \to \mathbf{Y}$. This extension is defined by the formula $L(\mathbf{x}) = L_0(\mathbf{x}^+) - L_0(\mathbf{x}^-)$, where $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-,\ \mathbf{x}^+, \mathbf{x}^- \in \mathbf{X}^+$.

Let $\mathcal{B}$ be the Borel σ-algebra of subsets of the set T. Let $\mathbf{m}\colon \mathcal{B} \to \mathbb{L}$ be *an additive regular operator valued measure*, i.e.

(i) $E, F \in \mathcal{B}, E \cap F = \emptyset \implies \mathbf{m}(E\cup F) = \mathbf{m}(E) + \mathbf{m}(F)$,

(ii) $\forall E \in \mathcal{B}, \forall \mathbf{x} \in \mathbf{X}^+, \exists \pi \in \mathbf{Y}^+, \forall \varepsilon > 0, \exists C \in \mathcal{B}$ (a compact set), $\exists O \in \mathcal{B}$ (an open set), such that $C \subset E \subset O$ and $\mathbf{m}(O\setminus C)\mathbf{x} < \varepsilon\pi$.

For $\rho \in \mathbf{X}^+, \pi > 0, \pi \in \mathbf{Y}$, we denote by $\|\mathbf{m}(E)\|_{\rho,\pi} = \|\mathbf{m}(E)\rho\|_\pi$, where $E \in \mathcal{B}$ and $\|\cdot\|_\pi$ is the Minkowski functional of $[-\pi,\pi]$ in $\mathbf{Y}(\pi)$.

Example 1.3. Let $(\mathcal{S},\Sigma_\mathcal{S},\nu), (\mathcal{V},\Sigma_\mathcal{V},\mu), (\mathcal{Z},\Sigma_\mathcal{Z},\varphi)$ be measure spaces with σ-finite nonegative measures, $(\mathcal{R},\Sigma_\mathcal{R},\lambda)$ be the measure product of those spaces, i.e. $\lambda = \varphi\otimes\nu\otimes\mu$ is a product measure on the generated σ-algebra $\Sigma_\mathcal{R} = \Sigma_\mathcal{Z}\otimes\Sigma_\mathcal{S}\otimes\Sigma_\mathcal{V}$ on $\mathcal{R} = \mathcal{Z}\times\mathcal{S}\times\mathcal{V}$. Let $\mathbf{X}, \mathbf{Y}$ be two spaces of measurable functions on $(\mathcal{S},\Sigma_\mathcal{S},\nu), (\mathcal{V},\Sigma_\mathcal{V},\mu)$, respectively, $\mathbf{X}^+ = \{\mathbf{x} \in \mathbf{X}; \forall s \in \mathcal{S} : \mathbf{x}(s) \geq 0\}, \mathbf{Y}^+ = \{\mathbf{y} \in \mathbf{Y}; \forall v \in \mathcal{V} : \mathbf{y}(v) \geq 0\}$. Let $\varkappa : \mathcal{Z}\times\mathcal{S}\times\mathcal{V} \to \mathbb{R}$ be a λ-measurable function. Then for every $\mathbf{x} \in \mathbf{X}$,

$$(\mathbf{m}(E)\mathbf{x})(v) = \int_E \int_\mathcal{S} \varkappa(z,s,v)\mathbf{x}(s)\ \mathrm{d}\nu(s)\ \mathrm{d}\varphi(z), E \in \Sigma_\mathcal{Z}$$

defines an integral operator and $(\mathbf{m}(E)\mathbf{x})(v) \geq 0$ for every $\mathbf{x} \in \mathbf{X}^+$ if and only if $\varkappa(z,s,v) \geq 0\ \ \lambda$-a.e., $(z,s,v) \in \mathcal{Z}\times\mathcal{S}\times\mathcal{V}$ (we consider Lebesgue integrals). (For the proof see [2], XI., p. 393.)

2. Definition of integral

A function $U : T \to 2^T$ is said to be *a neighborhood function* if for every $t \in T$, $U(t)$ is a neighborhood of t. By $\mathcal{U}$ we will denote a family of neighborhood functions (it will be specified in Definition 2.2) and by $\mathcal{U}_E$ the set of all restrictions $U|_{cl(E)}$ of functions $U \in \mathcal{U}$.

Definition 2.1. By *a partition* of $E \in \mathcal{B}$ we mean a finite set w of couples $\{(E_j, t_j); t_j \in cl(E_j), \bigcup_{j=1}^{J} E_j = E, E_i \cap E_j = \emptyset, E_i, E_j \in \mathcal{B}, i \neq j, i, j = 1, 2, \ldots, J\}$. For $U \in \mathcal{U}_E$, we will denote by $\mathcal{W}_E(U)$ the family of all partitions w of E such that $E_j \subset U(t_j), j = 1, 2, \ldots, J$.

Let $w = \{(E_i, t_i); i = 1, 2, \ldots, I\} \in \mathcal{W}_E(U), p = \{(F_j, z_j); j = 1, 2, \ldots, J\} \in \mathcal{W}_E(U), U \in \mathcal{U}_E$. We say that p is *a refinement* of w if each $E_i, i = 1, 2, \ldots, I$, is a union of some members of $\{F_j; j = 1, 2, \ldots, J\}$ and $\{t_i; i = 1, 2, \ldots, I\} \subset \{z_j; j = 1, 2, \ldots, J\}$.

Definition 2.2. A family $\mathcal{U} \neq \emptyset$ of neighborhood functions is said to be *satisfactory* (with respect to the set system $\mathcal{B}$ and the topology τ on T) if $\mathcal{W}_E(U) \neq \emptyset$ for every $E \in \mathcal{B}$ and $U \in \mathcal{U}$.

Example 2.3. The family of all neighborhood functions $\mathcal{U}$ is a satisfactory family for T the compact topological space and the $\mathbb{L}$-valued regular Borel measure defined in Section 1 of this paper (can be proved analogously as Lemma 3.4 in [8]). However in general, $\mathcal{U}$ need not be the family of all neighborhood functions and $\mathcal{B}$ the Borel σ-algebra. Particularly, when considering connections of the notion of integral and various types of integration bases, cf. [14], we should take the corresponding families $\mathcal{U}$ of neighborhood functions and the set systems $\mathcal{B}$. For instance, in case of the classical Riemann integral, $E = [a, b], \mathcal{U} = \{U(t) = (t-r; t+r); r > 0\}$ and $\mathcal{B} = \{F \subset T; F = A \setminus N\}$, where $a \leq b,\ a, b, r, t \in \mathbb{R},\ A$ is a finite interval and N the set of the Jordan measure zero on $\mathbb{R}$.

Definition 2.4. Let a family $\mathcal{U}$ of neighborhood functions be satisfactory. Let $E \in \mathcal{B}$. A function $\mathbf{f} : T \to \mathbf{X}$ is said to be π-$(\mathcal{U}, E)$-*integrable*, $\pi \in \mathbf{Y}^+$, if there exists $\mathbf{y} \in \mathbf{Y}$, such that

$$\forall \varepsilon > 0, \exists U \in \mathcal{U}_E, \forall w \in \mathcal{W}_E(U) : \left|\mathfrak{S}_{(w,U)}(\mathbf{f}, E) - \mathbf{y}\right| < \varepsilon\pi, \tag{1}$$

where $\mathfrak{S}_{(w,U)}(\mathbf{f}, E) = \sum_{j=1}^{J} \mathbf{m}(E_j)\mathbf{f}(t_j), w \in \mathcal{W}_E(U), (E_j, t_j) \in w, j = 1, 2, \ldots, J$. A function $\mathbf{f} : T \to \mathbf{X}$ is said to be $(\mathcal{U}, E)$-*integrable*, if there exists $\pi \in \mathbf{Y}^+$, such that $\mathbf{f}$ is π-$(\mathcal{U}, E)$-*integrable*. The value $\mathbf{y}$ will be called a $(\mathcal{U}, E)$-*integral* of the function $\mathbf{f}$ and denoted by $\mathbf{y} = \int_E \mathbf{f}\ \ \mathrm{d}\mathbf{m}$. The class of all $(\mathcal{U}, E)$-integrable functions will be denoted by $\mathcal{I}(\mathcal{U}, E)$.

In what follows, we suppose $\mathcal{U}$ to be a satisfactory family of neighborhood functions $U : T \to 2^T$.

Lemma 2.5. *Let $E \in \mathcal{B}, \mathbf{f} \in \mathcal{I}(\mathcal{U}, E)$. Then the integral $\mathbf{y} = \int_E \mathbf{f}\ \ \mathrm{d}\mathbf{m}$ in Definition 2.4 is defined uniquely.*

Proof. Trivial.

Lemma 2.6. *Let $E \in \mathcal{B}$. If $\mathbf{f}, \mathbf{g}$ are $(\mathcal{U}, E)$-integrable functions, $\mathbf{h} = \lambda\mathbf{f} + \mathbf{g}, \lambda \in \mathbb{R}$, then $\mathbf{h}$ is a $(\mathcal{U}, E)$-integrable function and*

$$\int_E \mathbf{h} \quad \mathrm{d}\mathbf{m} = \lambda \int_E \mathbf{f} \quad \mathrm{d}\mathbf{m} + \int_E \mathbf{g} \quad \mathrm{d}\mathbf{m}. \tag{2}$$

Proof. $\mathfrak{S}_{(w,U)}(\mathbf{h}, E) = \lambda\mathfrak{S}_{(w,U)}(\mathbf{f}, E) + \mathfrak{S}_{(w,U)}(\mathbf{g}, E)$.

Lemma 2.7. *Let $E \in \mathcal{B}$. The $(\mathcal{U}, E)$-integral is a positive operator, i.e. if $\mathbf{g} \leq_E \mathbf{f}$, $\mathbf{g}, \mathbf{f} \in \mathcal{I}(\mathcal{U}, E)$, then $\int_E \mathbf{g} \quad \mathrm{d}\mathbf{m} \leq \int_E \mathbf{f} \quad \mathrm{d}\mathbf{m}$, where $\mathbf{g} \leq_E \mathbf{f}$ if and only if $\mathbf{g}(t) \leq \mathbf{f}(t)$ for every $t \in cl(E)$.*

Proof. It is enough to show the implication $0 \leq_E \mathbf{f} \Rightarrow 0 \leq \int_E \mathbf{f} \quad \mathrm{d}\mathbf{m}$. This follows from the implication $0 \leq_E \mathbf{f} \Rightarrow 0 \leq \mathfrak{S}_{(w,U)}(\mathbf{f}, E)$.

Lemma 2.8. *Let $\mathbf{m}$ be an additive regular $\mathbb{L}$-valued measure defined on the σ-algebra $\mathcal{B}$ of Borel subsets of T. Let $E \in \mathcal{B}$. If $\mathbf{f} = \sum_{j=1}^{J} \chi_{E_j}\mathbf{x}_j$ is a simple function and $\mathcal{U}$ is a satisfactory family, then $\mathbf{f} \in \mathcal{I}(\mathcal{U}, E)$ and $\int_E \mathbf{f} \quad \mathrm{d}\mathbf{m} = \sum_{j=1}^{J} \mathbf{m}(E_j \cap E)\mathbf{x}_j$.*

Proof. Same as in [9], Th. 8.

Lemma 2.9. *Let $\mathbf{Y}$ be a complete vector lattice, $E \in \mathcal{B}$ and the class $\mathcal{U}$ be satisfactory. Then the following two assertions are equivalent:*

(i) $\mathbf{f} \in \mathcal{I}(\mathcal{U}, E)$,
(ii) $\exists \pi \in \mathbf{Y}^+, \forall \varepsilon > 0, \exists U \in \mathcal{U}_E, \forall w_1, w_2 \in \mathcal{W}_E(U)$:

$$\left|\mathfrak{S}_{(w_1,U)}(\mathbf{f}, E) - \mathfrak{S}_{(w_2,U)}(\mathbf{f}, E)\right| < \varepsilon\pi.$$

Proof. Same as in [9], Lemma 6.

Lemma 2.10. *Let $\mathbf{Y}$ be a complete vector lattice. If $E, F, G \in \mathcal{B}, E = F \cup G, F \cap G = \emptyset$, then*

(i) $\mathcal{I}(\mathcal{U}, E) \subset \mathcal{I}(\mathcal{U}, F) \cap \mathcal{I}(\mathcal{U}, G)$,
(ii) *for every* $\mathbf{f} \in \mathcal{I}(\mathcal{U}, E)$,

$$\int_E \mathbf{f} \quad \mathrm{d}\mathbf{m} = \int_F \mathbf{f} \quad \mathrm{d}\mathbf{m} + \int_G \mathbf{f} \quad \mathrm{d}\mathbf{m}. \tag{3}$$

Proof. (i) Take $\mathbf{f} \in \mathcal{I}(\mathcal{U}, E)$. By Definition 2.4, there exists $\pi \in \mathbf{Y}^+$ such that

$$\forall \varepsilon > 0, \exists U_E \in \mathcal{U}_E, \forall w \in \mathcal{W}_E(U_E) : \left|\mathfrak{S}_{(w,U_E)}(\mathbf{f}, E) - \int_E \mathbf{f} \quad \mathrm{d}\mathbf{m}\right| < \varepsilon\pi.$$

Then

$$|\mathfrak{S}_{(w_1,U_E)}(\mathbf{f},E) - \mathfrak{S}_{(w_2,U_E)}(\mathbf{f},E)| < 2\varepsilon \cdot \pi$$

for every $w_1, w_2 \in \mathcal{W}_E(U_E)$. Take $w', w'' \in \mathcal{W}_F(U_F)$ and $w_0 \in \mathcal{W}_G(U_G)$, where $U_F = U_E|_{cl(F)}, U_G = U_E|_{cl(G)}$. Put $w_1 = w' \cup w_0, w_2 = w'' \cup w_0$. Then $w_1, w_2 \in \mathcal{W}_E(U_E)$ and $|\mathfrak{S}_{(w_1,U_E)}(\mathbf{f},E) - \mathfrak{S}_{(w_2,U_E)}(\mathbf{f},E)| < \varepsilon \cdot \pi$. But

$$\begin{aligned}
|\mathfrak{S}_{(w',U_F)}(\mathbf{f},F) - \mathfrak{S}_{(w'',U_F)}(\mathbf{f},F)| = \\
&= |\mathfrak{S}_{(w',U_F)}(\mathbf{f},F) + \mathfrak{S}_{(w_0,U_G)}(\mathbf{f},G) - \\
&- \mathfrak{S}_{(w_0,U_G)}(\mathbf{f},G) - \mathfrak{S}_{(w'',U_F)}(\mathbf{f},F)| \\
&= |\mathfrak{S}_{(w_1,U_E)}(\mathbf{f},E) - \mathfrak{S}_{(w_2,U_E)}(\mathbf{f},E)| < 2\varepsilon \cdot \pi
\end{aligned}$$

for every $w', w'' \in \mathcal{W}_F(U_F)$. By Lemma 2.9, $\mathbf{f}$ is $(\mathcal{U}, F)$-integrable. Similarly, $\mathbf{f} \in \mathcal{I}(\mathcal{U}, G)$. Hence, $\mathcal{I}(\mathcal{U}, E) \subset \mathcal{I}(\mathcal{U}, F) \cap \mathcal{I}(\mathcal{U}, G)$.

(ii) Let $w \in \mathcal{W}_E(U_E)$ be an arbitrary partition. Denote by $w_{F\cup G}$ a refinement partition of w, such that $w_F \cup w_G = w_{F\cup G}$, where $w_F \in \mathcal{W}_F(U_F), w_G \in \mathcal{W}_G(U_G)$. Then $w_{F\cup G} \in \mathcal{W}_E(U_E)$ and

$$\left|\mathfrak{S}_{(w_{F\cup G},U_E)}(\mathbf{f},E) - \int_E \mathbf{f} \ \ \mathrm{d}\mathbf{m}\right| < \varepsilon\pi.$$

Now the equality (3) follows from (i) and the equality

$$\mathfrak{S}_{(w_{F\cup G},U_E)}(\mathbf{f},E) = \mathfrak{S}_{(w_F,U_F)}(\mathbf{f},E) + \mathfrak{S}_{(w_G,U_G)}(\mathbf{f},E).$$

3. Convergence theorem

To prove a convergence theorem for our integral, Theorem 3.3, we need two usefull preparatory lemmas.

Lemma 3.1 (R. Henstock, J. Kurzweil). *Let* $\mathbf{Y}$ *be a complete vector lattice. Let* $E \in \mathcal{B}, \mathbf{f} \in \mathcal{I}(\mathcal{U}, E), \varepsilon > 0, U \in \mathcal{U}_E, \pi \in \mathbf{Y}^+$. *If* $w \in \mathcal{W}_E(U), w = \{(E_j, t_j); j = 1, 2, \dots, J\}$,

$$\left|\int_E \mathbf{f} \ \ \mathrm{d}\mathbf{m} - \mathfrak{S}_{(w,U)}(\mathbf{f},E)\right| < \varepsilon\pi$$

and $A \subset \{1, 2, \dots, J\}$, *then*

$$\left|\sum_{j\in A} \int_{E_j} \mathbf{f} \ \ \mathrm{d}\mathbf{m} - \sum_{j\in A} \mathbf{m}(E_j)\mathbf{f}(t_j)\right| \le \varepsilon\pi.$$

Proof. By lemma 2.10, $\mathbf{f} \in \mathcal{I}(\mathcal{U}_{E_j}, E_j)$. Choose $w_j \in \mathcal{W}_{E_j}(U_{E_j})$ such that

$$\left| \int_{E_i} \mathbf{f} \ \ \mathrm{d}\mathbf{m} - \mathfrak{S}_{(w_j, U_{E_j})}(\mathbf{f}, E_j) \right| < \frac{\eta}{2^j}\pi,$$

where η is an arbirtary positive number, $j \in \{1, 2, \dots, J\} \setminus A = B$. (From the proof of Lemma 2.10 we see that we can take the same regularizator π.) Put

$$w^* = \{(E_j, t_j) : j \in A\} \cup \bigcup_{j \in B} w_j.$$

Since $w^* \in \mathcal{W}_E(U)$,

$$\left| \int_{E} \mathbf{f} \ \ \mathrm{d}\mathbf{m} - \mathfrak{S}_{(w^*, U)}(\mathbf{f}, E) \right| < \varepsilon\pi.$$

We have: $\mathfrak{S}_{(w^*,U)}(\mathbf{f}, E) = \sum_{j \in A} \mathbf{m}(E_j)\mathbf{f}(t_j) + \sum_{j \in B} \mathbf{m}(E_j)\mathbf{f}(t_j)$. Therefore

$$\left| \sum_{j \in A} \int_{E_j} \mathbf{f} \ \ \mathrm{d}\mathbf{m} - \sum_{j \in A} \mathbf{m}(E_j)\mathbf{f}(t_j) \right| \leq$$

$$\leq \left| \int_{E} \mathbf{f} \ \ \mathrm{d}\mathbf{m} - \mathfrak{S}_{(w^*, U)}(\mathbf{f}, E) \right| + \left| \sum_{j \in B} \int_{E_j} \mathbf{f} \ \ \mathrm{d}\mathbf{m} - \sum_{j \in B} \mathbf{m}(E_j)\mathbf{f}(t_j) \right|$$
$$< (\varepsilon + \eta)\pi.$$

Since the last inequality holds for every $\eta > 0$, we obtain the desired property.

Lemma 3.2. *Let $E \in \mathcal{B}$. Let $\mathbf{f} \in \mathcal{I}(\mathcal{U}, E)$. Let there exist $\rho \in \mathbf{X}, \rho > 0; \pi \in \mathbf{Y}, \pi > 0$, such that $\|\mathbf{f}\|_{E,\rho} < \infty, \|\mathbf{m}(E)\|_{\rho,\pi} < \infty$. Then*

$$\left| \int_{E} \mathbf{f} \ \ \mathrm{d}\mathbf{m} \right| \leq \|\mathbf{f}\|_{E,\rho} \cdot \|\mathbf{m}(E)\|_{\rho,\pi} \cdot \pi. \tag{4}$$

Proof. The cases $\|\mathbf{f}\|_{E,\rho} = 0$ and $\|\mathbf{m}(E)\|_{\rho,\pi} = 0$ are trivial. Let $0 < \|\mathbf{f}\|_{E,\rho}, 0 < \|\mathbf{m}(E)\|_{\rho,\pi}$. Let $\varepsilon > 0, U \in \mathcal{U}_E, w \in \mathcal{W}_E(U)$.

Observe that if $|\mathbf{x}_j| \leq \rho$, and $E_j \in \mathcal{B}$ are pairwise disjoint, $j = 1, 2, \dots, J$, $\bigcup_{j=1}^J E_j = E$, then

$$\left| \sum_{j=1}^J \mathbf{m}(E_j)\mathbf{x}_j \right| \leq \left| \sum_{j=1}^J \mathbf{m}(E_j)\rho \right| =$$

$$= \left| \left(\sum_{j=1}^J \mathbf{m}(E_j) \right) \rho \right| = |\mathbf{m}(E)\rho| \leq \|\mathbf{m}(E)\rho\|_\pi \cdot \pi = \|\mathbf{m}(E)\|_{\rho,\pi} \cdot \pi.$$

Then the inequality $\left|\int_E \mathbf{f} \ \ \mathrm{d}\mathbf{m} - \mathfrak{S}_{(w,U)}(\mathbf{f},E)\right| < \varepsilon \cdot \pi$ in (1) implies

$$\left|\int_E \mathbf{f} \ \ \mathrm{d}\mathbf{m}\right| \le \left|\int_E \mathbf{f} \ \ \mathrm{d}\mathbf{m} - \mathfrak{S}_{(w,U)}(\mathbf{f},E)\right| + |\mathfrak{S}_{(w,U)}(\mathbf{f},E)| <$$
$$< \varepsilon\pi + |\mathfrak{S}_{(w,U)}(\mathbf{f},E)| = \varepsilon\pi + \left|\sum_{j=1}^{J} \mathbf{m}(E_j)\frac{\mathbf{f}(t_j)}{\|\mathbf{f}\|_{E,\rho}}\right| \cdot \|\mathbf{f}\|_{E,\rho} \le$$
$$\le \varepsilon\pi + \|\mathbf{f}\|_{E,\rho} \cdot \|\mathbf{m}(E)\|_{\rho,\pi} \cdot \pi = (\varepsilon + \|\mathbf{f}\|_{E,\rho} \cdot \|\mathbf{m}(E)\|_{\rho,\pi}) \cdot \pi.$$

Since the last inequality holds for every $\varepsilon > 0$, we obtain the desired assertion.

Theorem 3.3. *Let $\mathbf{X}$ be an Archimedean vector lattice and $\mathbf{Y}$ be an almost regular complete vector lattice. Let $\mathbf{m}$ be an additive regular $\mathbb{L}$-valued measure defined on the σ-algebra $\mathcal{B}$ of Borel subsets of T. Let the family $\mathcal{U}$ of neighborhood functions be satisfactory. Let $E \in \mathcal{B}$. If there exists a sequence $(\mathbf{f}_i)_{i\in\mathbb{N}}$ of $(\mathcal{U},E)$-integrable functions, such that*

(i) $\mathbf{f}_i \le_E \mathbf{f}_{i+1}, i \in \mathbb{N}$,

(ii) *there exists a function $\mathbf{f} : T \to \mathbf{X}$, such that*

$$(r)\text{-}\lim_{i\to\infty} \mathbf{f}_i(t) = \mathbf{f}(t) \tag{5}$$

with the same regularizator $\rho \in \mathbf{X}^+$ for every $t \in cl(E)$,

(iii) *there exists $\pi \in \mathbf{Y}^+$, such that for every $i \in \mathbb{N}$, the functions $\mathbf{f}_i$ are π-$(\mathcal{U},E)$-integrable and*

$$\int_E \mathbf{f}_i \ \ \mathrm{d}\mathbf{m} \le \pi, \tag{6}$$

(iv) $\mathbf{m}(E)\rho \in \mathbf{Y}(\pi)$,

then $\mathbf{f} \in \mathcal{I}(\mathcal{U},E)$ and $\int_E \mathbf{f} \ \ \mathrm{d}\mathbf{m} = (r)\text{-}\lim_{i\to\infty} \int_E \mathbf{f}_i \ \mathrm{d}\mathbf{m}$.

Proof. The cases $\rho = 0, \pi = 0$ are trivial. Suppose $\rho > 0, \pi > 0$. The plan of the proof is the following. First we show that the limit $(r)\text{-}\lim_{i\to\infty} \int_E \mathbf{f}_i \ \mathrm{d}\mathbf{m} = \mathbf{y}$ exists. To given $\varepsilon > 0$ we construct $U \in \mathcal{U}_E$. Then we take an arbitrary partition $w = \{(E_j,t_j); j = 1,2,\dots,J\} \in \mathcal{W}_E(U)$ and show that $|\mathfrak{S}_{(w,U)}(\mathbf{f},E) - \mathbf{y}| < (2 + \|\mathbf{m}(E)\|_{\rho,\pi}) \cdot \varepsilon\pi$, i.e. $\int_E \mathbf{f} \ \ \mathrm{d}\mathbf{m} = \mathbf{y}$.

Step 1. (Existence of the limit.)

Consider the (ρ)-ideal in $\mathbf{X}$ and the (π)-ideal in $\mathbf{Y}$. Let $\varepsilon > 0$ be given.

The inequality (6) implies that the sequence $\left(\int_E \mathbf{f}_i \ \ \mathrm{d}\mathbf{m}\right)_{i\in\mathbb{N}}$, is bounded. Since the vector lattice $\mathbf{Y}$ is complete, there exists $\mathbf{y} \in \mathbf{Y}$, such that

$\mathbf{y} = \bigvee_{i=1}^{\infty} \mathbf{y}_i$, where $\mathbf{y}_i = \int_E \mathbf{f}_i \ \mathrm{d}\mathbf{m}$. The assumption (iii) implies $\mathbf{y} \leq \pi$. By Lemma 2.9 and (ii), $(\mathbf{y}_i)_{i\in\mathbb{N}}$ is a nondecreasing sequence of elements in $\mathbf{Y}$ and hence, it (o)-converges to $\mathbf{y}$, cf. [13], Th. II.6.1. Since the complete vector lattice $\mathbf{Y}$ is almost regular, the sequence $(\mathbf{y}_i)_{i\in\mathbb{N}}$ of integrals (r)-converges to $\mathbf{y}$ with a regularizator $\pi_1 \in \mathbf{Y}^+$. Without loss of generality suppose $\pi_1 = \pi$ (if not, deal with $\pi_1 \vee \pi$, which is a regularizator which satisfies (iv), too). So, there exists $i_1 \in \mathbb{N}$, such that for every $i \geq i_1, i \in \mathbb{N}$,

$$\left| \int_E \mathbf{f}_i \ \mathrm{d}\mathbf{m} - \mathbf{y} \right| < \varepsilon\pi. \tag{7}$$

So, $\mathbf{y} = (r)\text{-}\lim_{i\to\infty} \int_E \mathbf{f}_i \ \mathrm{d}\mathbf{m}$ (with the regularizator π).

Step 2. (Construction of the neighborhood function.)

Definition 2.4 implies that for every $\varepsilon > 0$ and $\mathbf{f}_i \in \mathcal{I}(\mathcal{U}, E), i \in \mathbb{N}$, there exists its $U^{(i)} \in \mathcal{U}_E$, such that $\mathcal{W}_E(U^{(i)}) \neq \emptyset$ and for every partition $w^{(i)} \in \mathcal{W}_E(U^{(i)})$,

$$\left| \int_E \mathbf{f}_i \ \mathrm{d}\mathbf{m} - \mathfrak{S}_{(w^{(i)},U^{(i)})}(\mathbf{f}_i, E) \right| < \frac{\varepsilon}{2^i}\pi. \tag{8}$$

From (5) we see that the following function $\varphi : cl(E) \to \mathbb{N}$ is defined correctly:

$$\varphi(t) = \min\left\{ i \in \mathbb{N}; |\mathbf{f}_i(t) - \mathbf{f}(t)| \leq \varepsilon\rho \quad \text{and} \quad i \geq i_1 \right\}, t \in cl(E). \tag{9}$$

Construct a function $U \in \mathcal{U}_E$ by the formula $U(t) = U^{(\varphi(t))}(t), t \in cl(E)$.

Step 3. (Splitting the inequality into three parts.)

Take an arbitrary partition $w \in \mathcal{W}_E(U)$. We have:

$$\begin{aligned}
|\mathfrak{S}_{(w,U)}(\mathbf{f}, E) - \mathbf{y}| \leq & \left| \sum_{j=1}^{J} \mathbf{m}(E_j)(\mathbf{f}(t_j) - \mathbf{f}_{\varphi(t_j)}(t_j)) \right| + \\
& + \left| \sum_{j=1}^{J} \left[\mathbf{m}(E_j)\mathbf{f}_{\varphi(t_j)}(t_j) - \int_{E_j} \mathbf{f}_{\varphi(t_j)} \ \mathrm{d}\mathbf{m} \right] \right| + \\
& + \left| \sum_{j=1}^{J} \int_{E_j} \mathbf{f}_{\varphi(t_j)} \ \mathrm{d}\mathbf{m} - \mathbf{y} \right|.
\end{aligned} \tag{10}$$

Step 4. (Using Lemma 3.2.)

Denote by $\mathbf{f}(t_j) - \mathbf{f}_{\varphi(t_j)}(t_j) = \mathbf{x}_j$ and $\mathbf{g} = \sum_{j=1}^{J} \chi_{E_j}\mathbf{x}_j, j = 1, 2, \dots, J$. By Lemma 2.8, $\mathbf{g} \in \mathcal{I}(\mathcal{U}, E)$. By (9), $|\mathbf{g}(t)| \leq \varepsilon\rho, t \in cl(E)$. Since $\|\cdot\|_\rho$ is a monotone norm in $\mathbf{X}(\rho)$, $\|\mathbf{g}(t)\|_\rho \leq \varepsilon, t \in cl(E)$. Thus, $\|\mathbf{g}\|_{E,\rho} \leq \varepsilon$. By Lemma 3.2 and (iv),

$$\left|\sum_{j=1}^{J} \mathbf{m}(E_j)(\mathbf{f}(t_j) - \mathbf{f}_{\varphi(t_j)}(t_j))\right| = \\ = \left|\int_E \mathbf{g} \ \ \mathrm{d}\mathbf{m}\right| \leq \|\mathbf{g}\|_{E,\rho} \cdot \|\mathbf{m}(E)\|_{\rho,\pi} \cdot \pi \leq \\ \leq \|\mathbf{m}(E)\|_{\rho,\pi} \cdot \pi\varepsilon.$$

Step 5. (Using Lemma 3.1.)

Put $s = \max\{\varphi(t_j); j = 1, 2, \dots, J\}$. Then by (8) and Lemma 3.1,

$$\left|\sum_{j=1}^{J} \left[\mathbf{m}(E_j)\mathbf{f}(t_j) - \int_{E_j} \mathbf{f}_{\varphi(t_j)} \ \ \mathrm{d}\mathbf{m}\right]\right| \leq \\ \leq \sum_{i=1}^{s} \left|\sum_{j:\varphi(t_j)=i} \left[\mathbf{m}(E_j)\mathbf{f}_i(t_j) - \int_{E_j} \mathbf{f}_i \ \ \mathrm{d}\mathbf{m}\right]\right| \leq \\ \leq \sum_{i=1}^{s} \frac{\varepsilon}{2^i} \cdot \pi < \varepsilon\pi.$$

Step 6. (Using Lemmas 2.7 and 2.10.)

Put $r = \min\{\varphi(t_j); j = 1, 2, \dots, J\}$, so that $s \geq r$. By (i) and Lemma 2.7, and Lemma 2.10,

$$\int_E \mathbf{f}_r \ \ \mathrm{d}\mathbf{m} = \sum_{j=1}^{J} \int_{E_j} \mathbf{f}_r \ \ \mathrm{d}\mathbf{m} \leq \\ \leq \sum_{j=1}^{J} \int_{E_j} \mathbf{f}_{\varphi(t_j)} \ \ \mathrm{d}\mathbf{m} \leq \\ \leq \sum_{j=1}^{J} \int_{E_j} \mathbf{f}_s \ \ \mathrm{d}\mathbf{m} = \int_E \mathbf{f}_s \ \ \mathrm{d}\mathbf{m} \leq \mathbf{y}.$$

Consequently, by (7),

$$\left|\sum_{j=1}^{J} \int_{E_j} \mathbf{f}_{\varphi(t_j)} \ \ \mathrm{d}\mathbf{m} - \mathbf{y}\right| \leq \left|\mathbf{y} - \int_E \mathbf{f}_r \ \ \mathrm{d}\mathbf{m}\right| \leq \varepsilon\pi.$$

The proof is complete.

Remark 3.4. Theorem 3.3 is a generalization (with slight modifications) of Theorems 5.1 in [7], 2 in [5], and 2.2 in [8].

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