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ON A LATTICE STRUCTURE OF OPERATOR SPACES IN COMPLETE BORNOLOGICAL LOCALLY CONVEX SPACES

JÁN HALUŠKA

ABSTRACT. For $\mathbf{X}, \mathbf{Y}$ complete bornological locally convex spaces, we consider a lattice structure of the space $L(\mathbf{X}, \mathbf{Y})$ of all continuous linear operators $L\colon \mathbf{X} \to \mathbf{Y}$.

INTRODUCTION

The description of theory of complete bornological locally convex spaces (C.B.L.C.S.) we can find in [4], [6], and [3].

In [1], [2] we have developed a technique for an operator valued measure $\mathbf{m} : \Delta \to L(\mathbf{X}, \mathbf{Y})$, where Δ is a δ-ring of sets, $L(\mathbf{X}, \mathbf{Y})$ the space of all continuous operators $L\colon \mathbf{X} \to \mathbf{Y}$, where $\mathbf{X}, \mathbf{Y}$ are both C.B.L.C.S. In [1] we gave a more detail explanation of basic $L(\mathbf{X}, \mathbf{Y})$-measure set structures (H. Weber, cf. [7], considered these structures particularly from topological aspects.). In connection with it, a Bartle type integral was investigated. In [2], convergences in measure, almost everywhere, almost uniform (and relations between them) were studied.

In the present paper we consider the lattice structure of the range space of such measure $\mathbf{m}$, the space $L(\mathbf{X}, \mathbf{Y})$.

1. PRELIMINARIES

Let $\mathbf{X}, \mathbf{Y}$ be two C.B.L.C.S. over the field of real or complex numbers equipped with the bornologies $\mathfrak{B}_{\mathbf{X}}, \mathfrak{B}_{\mathbf{Y}}$. The basis $\mathcal{U}$ of the bornology $\mathfrak{B}_{\mathbf{X}}$ has *a marked element* $u_0 \in \mathcal{U}$, if $u_0 \subset u$ for every $u \in \mathcal{U}$. Let the bases $\mathcal{U}, \mathcal{W}$ be chosen to consist of all $\mathfrak{B}_{\mathbf{X}}$-, $\mathfrak{B}_{\mathbf{Y}}$- bounded Banach disks in $\mathbf{X}, \mathbf{Y}$, with marked elements $u_0 \in \mathcal{U}, u_0 \neq \{0\}$, and $w_0 \in \mathcal{W}, w_0 \neq \{0\}$, respectively. Remind that *a Banach disk* in $\mathbf{X}$ is a set which is closed, absolutely convex and the linear span of which is a Banach space. The space $\mathbf{X}$ is an inductive limit of Banach spaces $\mathbf{X}_u, u \in \mathcal{U}$,

$$\mathbf{X} = \lim \operatorname{ind}_{u \in \mathcal{U}} \mathbf{X}_u,$$

cf. [4], where $\mathbf{X}_u$ is a linear span of $u \in \mathcal{U}$ and $\mathcal{U}$ is directed by inclusion (analogously for $\mathbf{Y}$ and $\mathcal{W}$).

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On $\mathcal{U}$ *the lattice operations* are defined as follows. For $u_1, u_2 \in \mathcal{U}$ we have: $u_1 \wedge u_2 = u_1 \cap u_2$, $u_1 \vee u_2 = \operatorname{acs}(u_1 \cup u_2)$, where acs denotes the topological closure of the absolutely convex span of the set. Analogously for $\mathcal{W}$. For $(u_1, w_1), (u_2, w_2) \in \mathcal{U} \times \mathcal{W}$, we write $(u_1, w_1) \ll (u_2, w_2)$ if and only if $u_1 \subset u_2$ and $w_1 \supset w_2$.

2. Lattice structure of $L(\mathbf{X}, \mathbf{Y})$

If p_w is Minkowski functional of the set $w \in \mathcal{W}$, then for $u \in \mathcal{U}, L \in L(\mathbf{X}, \mathbf{Y})$, we put $p_{u,w}(L) = \sup_{\mathbf{x} \in u} p_w(L(\mathbf{x}))$ (If w does not absorb $L(\mathbf{x}), \mathbf{x} \in u$, we put $p_{u,w}(L) = \infty$.). Denote by $\mathcal{L}_{u,w} = \{L \in L(\mathbf{X}, \mathbf{Y});\ p_{u,w}(L) < \infty\}, (u, w) \in \mathcal{U} \times \mathcal{W}$, and $\mathfrak{L}_{\mathcal{U},\mathcal{W}} = \{\mathcal{L}_{u,w};\ (u, w) \in \mathcal{U} \times \mathcal{W}\}$. For $(u, w) \in \mathcal{U} \times \mathcal{W}$, a sequence $L_n \in L(\mathbf{X}, \mathbf{Y}), n = 1, 2, \dots,$ is said to be convergent to $L \in L(\mathbf{X}, \mathbf{Y})$ in $\mathcal{L}_{u,w}$ whenever $\lim_{n \to \infty} p_{u,w}(L_n - L) = 0$.

On $\mathfrak{L}_{\mathcal{U},\mathcal{W}}$ define the operations $\wedge, \vee$ and *an order* $\ll$. For $(u_1, w_1), (u_2, w_2) \in \mathcal{U} \times \mathcal{W}$,

$$\mathcal{L}_{u_1,w_1} \vee \mathcal{L}_{u_2,w_2} = \mathcal{L}_{u_1 \wedge u_2, w_1 \vee w_2}, \mathcal{L}_{u_1,w_1} \wedge \mathcal{L}_{u_2,w_2} = \mathcal{L}_{u_1 \vee u_2, w_1 \wedge w_2},$$

$$\mathcal{L}_{u_2,w_2} \ll \mathcal{L}_{u_1,w_1} \quad \text{if and only if} \quad (u_1, w_1) \ll (u_2, w_2).$$

It is easy to see that $\wedge, \vee$ are *lattice operations*.

Theorem 1. *The family $\mathfrak{L}_{\mathcal{U},\mathcal{W}}$ of operator spaces is a distributive lattice.*

Proof. For $(u_1, w_1), (u_2, w_2), (u_3, w_3) \in \mathcal{U} \times \mathcal{W}$, we have:

$$\begin{aligned}
\mathcal{L}_{u_1,w_1} \vee (\mathcal{L}_{u_2,w_2} \wedge \mathcal{L}_{u_3,w_3}) &= \mathcal{L}_{u_1,w_1} \vee \mathcal{L}_{u_2 \vee u_3, w_2 \wedge w_3} \\
&= \mathcal{L}_{u_1 \wedge (u_2 \vee u_3), w_1 \vee (w_2 \wedge w_3)} \\
&= \mathcal{L}_{(u_1 \wedge u_2) \vee (u_1 \wedge u_3), (w_1 \vee w_2) \wedge (w_1 \vee w_3)} \\
&= \mathcal{L}_{u_1 \wedge u_2, w_1 \vee w_2} \wedge \mathcal{L}_{u_1 \wedge u_3, w_1 \vee w_2} \\
&= (\mathcal{L}_{u_1,w_1} \vee \mathcal{L}_{u_2,w_2}) \wedge (\mathcal{L}_{u_1,w_1} \vee \mathcal{L}_{u_3,w_2}).
\end{aligned}$$

By [5], Th.2.2, $\mathfrak{L}_{\mathcal{U},\mathcal{W}}$ is a distributive lattice.

The lattice $\mathfrak{L}_{\mathcal{U},\mathcal{W}}$ introduces a topology of an inductive limit on $L(\mathbf{X}, \mathbf{Y})$, i.e. there holds the following theorem.

Theorem 2. $L(\mathbf{X}, \mathbf{Y}) = \lim \operatorname{ind}_{(u,w) \in \mathcal{U} \times \mathcal{W}} \mathcal{L}_{u,w}$.

Proof. For $u \in \mathcal{U}, w \in \mathcal{W}$, it is easy to verify that $\mathcal{L}_{u,w}$ is a vector subspace of $L(\mathbf{X}, \mathbf{Y})$ equipped with the topology given by the seminorm $p_{u,w}$.

Show that $\bigcup_{(u,w) \in \mathcal{U} \times \mathcal{W}} \mathcal{L}_{u,w} = L(\mathbf{X}, \mathbf{Y})$. The inclusion $\bigcup_{(u,w) \in \mathcal{U} \times \mathcal{W}} \mathcal{L}_{u,w} \subset L(\mathbf{X}, \mathbf{Y})$ is trivial. Show $\bigcup_{(u,w) \in \mathcal{U} \times \mathcal{W}} \mathcal{L}_{u,w} \supset L(\mathbf{X}, \mathbf{Y})$. Let $L \in L(\mathbf{X}, \mathbf{Y})$. So, to each $u \in \mathcal{U}$ there exists $w_{u,L} \in \mathcal{W}$ such that $L(u) \subset w_{u,L}$, i.e. $p_{u,w_{u,L}}(L) \leq 1 < \infty$. Thus, $L \in \mathcal{L}_{u,w_{u,L}} \subset \bigcup_{(u,w) \in \mathcal{U} \times \mathcal{W}} \mathcal{L}_{u,w}$.

Let $(u_1, w_1), (u_2, w_2) \in \mathcal{U} \times \mathcal{W}$. Show now that if $\mathcal{L}_{u_2,w_2} \ll \mathcal{L}_{u_1,w_1}$, then $\mathcal{L}_{u_2,w_2} \subset \mathcal{L}_{u_1,w_1}$ and if a sequence $L_n \in L(\mathbf{X}, \mathbf{Y}), n = 1, 2, \dots,$ of operators converges to $L \in L(\mathbf{X}, \mathbf{Y})$ in $\mathcal{L}_{u_2,w_2}$, then it converges to L also in $\mathcal{L}_{u_1,w_1}$. Indeed, by definition, $(u_1, w_1) \ll (u_2, w_2) \Leftrightarrow u_1 \subset u_2 \wedge w_1 \supset w_2$. The relation $u_1 \subset u_2$ implies $p_{u_1,w}(L) \leq p_{u_2,w}(L)$ for every $w \in \mathcal{W}$. The inclusion $w_2 \subset w_1$ implies $p_{w_1}(L(\mathbf{x})) \leq p_{w_2}(L(\mathbf{x}))$

for every $\mathbf{x} \in \mathbf{X}$. From this we have $p_{u,w_1}(L)) \leq p_{u,w_2}(L)$ for every $u \in \mathcal{U}$. Thus, $p_{u_1,w_1}(L) \leq p_{u_1,w_2}(L) \leq p_{u_2,w_2}(L)$. So, if $(u_1, w_1) \ll (u_2, w_2)$ and $L \in L(\mathbf{X}, \mathbf{Y})$, then $p_{u_1,w_1}(L) \leq p_{u_2,w_2}(L)$. This completes the proof.

Note that in the terminology of [6], $L(\mathbf{X}, \mathbf{Y})$ (as an inductive limit of seminormed spaces) is *a bornological convex vector space*, cf. [6], chap. 4, §2, Th. 1.

Theorem 3. *For every $(u_1, w_1) \in \mathcal{U} \times \mathcal{W}$, the set*

$$\mathfrak{I}_{u_1,w_1} = \{\mathcal{L}_{u,w} \in \mathfrak{L}_{\mathcal{U},\mathcal{W}};\ \mathcal{L}_{u,w} \ll \mathcal{L}_{u_1,w_1},\ (u, w) \in \mathcal{U} \times \mathcal{W}\}$$

is an ideal in $\mathfrak{L}_{\mathcal{U},\mathcal{W}}$.

Proof. Let $(p, q), (u, w) \in \mathcal{U} \times \mathcal{W}$. and $(u_1, w_1) \ll (u, w), (u_1, w_1) \ll (p, q)$. Since $u \wedge p = u \cap p \supset u_1, w \vee q = \operatorname{acs}(w \cup q) \subset w_1$, then $\mathcal{L}_{u,w} \vee \mathcal{L}_{p,q} = \mathcal{L}_{u\wedge p,w\vee q} \ll \mathcal{L}_{u_1,w_1}$.

Let $(p, q), (u, w) \in \mathcal{U} \times \mathcal{W}$, and $(u_1, w_1) \ll (p, q)$. Then $\mathcal{L}_{u,w} \wedge \mathcal{L}_{p,q} = \mathcal{L}_{u\vee p,w\wedge q} \ll \mathcal{L}_{u_1,w_1}$.

Dually to Theorem 3, we obtain the following corollary.

Corollary 4. For every $(u_2, w_2) \in \mathcal{U} \times \mathcal{W}$, the set

$$\mathfrak{F}_{u_2,w_2} = \{\mathcal{L}_{u,w} \in \mathfrak{L}_{\mathcal{U},\mathcal{W}};\ \mathcal{L}_{u_2,w_2} \ll \mathcal{L}_{u,w},\ (u, w) \in \mathcal{U} \times \mathcal{W}\},$$

is a filter in $\mathfrak{L}_{\mathcal{U},\mathcal{W}}$.

Theorem 5. *Let $(u_1, w_1), (u_2, w_2) \in \mathcal{U} \times \mathcal{W}$. If $(u_1, w_1) \ll (u_2, w_2)$, then the order interval $[\mathcal{L}_{u_2,w_2}, \mathcal{L}_{u_1,w_1}] = \mathfrak{I}_{u_1,w_1} \cap \mathfrak{F}_{u_2,w_2}$ in $\mathfrak{L}_{\mathcal{U},\mathcal{W}}$ is a Boolean algebra with $\mathcal{L}_{u_2,w_2}$ as null and $\mathcal{L}_{u_1,w_1}$ as unit.*

Proof. Let $(u, w) \in \mathcal{U} \times \mathcal{W}, (u_1, w_1) \ll (u, w) \ll (u_2, w_2)$. Put

$$\mathcal{L}^{\perp}_{u,w} = \mathcal{L}_{(u_2\setminus u)\vee u_1,(w_1\setminus w)\vee w_2} \in [\mathcal{L}_{u_2,w_2}, \mathcal{L}_{u_1,w_1}]$$

and show that $\mathcal{L}^{\perp}_{u,w}$ is a complement of $\mathcal{L}_{u,w}$ in $[\mathcal{L}_{u_2,w_2}, \mathcal{L}_{u_1,w_1}]$. We have:

$$\begin{aligned}\mathcal{L}_{u,w} \vee \mathcal{L}^{\perp}_{u,w} =& \mathcal{L}_{u,w} \vee \mathcal{L}_{(u_2\setminus u)\vee u_1,(w_1\setminus w)\vee w_2}\\ =&\mathcal{L}_{u\wedge[(u_2\setminus u)\vee u_1],w\vee[(w_1\setminus w)\vee w_2]}\\ =&\mathcal{L}_{[u\wedge(u_2\setminus u)]\vee[u\wedge u_1],w_1\vee w_2}\\ =&\mathcal{L}_{u_1,w_1}.\end{aligned}$$

Analogously, $\mathcal{L}_{u,w} \wedge \mathcal{L}^{\perp}_{u,w} = \mathcal{L}_{u_2,w_2}$. So, $\mathcal{L}_{u_2,w_2}$ is the null and $\mathcal{L}_{u_1,w_1}$ is the unit of the Boolean algebra $[\mathcal{L}_{u_2,w_2}, \mathcal{L}_{u_1,w_1}]$.

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Mathematical Institute, Slovak Academy of Sciences, Grešákova 6, 040 01 Košice, Slovakia
E-mail address: `jhaluska @ saske.sk`