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ON LATTICES OF SET FUNCTIONS IN COMPLETE BORNOLOGICAL LOCALLY CONVEX SPACES

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ABSTRACT. For complete bornological locally convex spaces $\mathbf{X}, \mathbf{Y}$, various natural lattices of set functions and families of sets related to an $L(\mathbf{X}, \mathbf{Y})$-valued measure are introduced. In connection with it, a Bartle-type integral is investigated.

INTRODUCTION

H. Weber in [20] and [21] deals with the lattice of all s-bounded monotone ring topologies (= FN-topologies) on a Boolean ring. Besides criteria for completeness and metrizability he obtains the decomposition theorems for monotone ring topologies. The theory is applied to the problem of Lebesgue decomposition. His approach is rather algebraic.

Our aim in this paper is somewhat different. Let $\mathbf{X}$ and $\mathbf{Y}$ be complete bornological locally convex spaces (for basic notions of bornology theory see the monographs by J. V. Radyno, [18], H. Jarchow, [13], and H. Hogbe-Nlend, [12]). We introduce various natural lattices of set functions and families of sets related to an $L(\mathbf{X}, \mathbf{Y})$-valued measure.

This serves as a background for several types of $L(\mathbf{X}, \mathbf{Y})$-valued measure integrations. In particular, the final section of this paper presents an application of developed techniques to the generalization of a Bartle-type integration theory in complete bornological locally convex spaces.

Roughly spoken, there are at least two reasons why the extension of Bartle's integration theory, cf. [2], to locally convex spaces is not so easy: (i) In locally convex spaces net convergences are typical and, in general, for nets of functions Jegorov's theorem does not hold, cf. e.g. Th. 3.4. in [11]. (ii) For non-metrizable linear spaces, there is a non-trivial question what is a null set and, consequently, also "the convergence of functions in measure". We will see that the majority of notions, simple in the classical theory, cf. e.g. [9] and also in the theories of H. Weber and R. G. Bartle, will be replaced in our theory by lattices. For instance, we will work with lattices of set functions, set systems, null sets, bornologies.

Remind that each complete bornological locally convex space is an inductive limit of separable Banach spaces and the convergence on it is the so called bornological convergence which is also a sequential convergence in this case, cf. [3]. The bornological convergence, when the bornology is von Neumann (a set is von Neumann bounded iff it is absorbed by every zero-neighborhood), implies the topological convergence. On the other hand, we can introduce the von Neumann bornology on an arbitrary complete locally convex space $\mathbf{X}$ and the topological completeness of $\mathbf{X}$ implies the completeness in the sense of the bornology, see J.V.Radyno, [18], chap. 4., §5, Prop.5. In this paper we will handle the difficulties in developing the general theory on integration in locally convex spaces mentioned above only in the

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special case of complete bornological locally convex vector spaces. This makes it possible to define various integrals in the general case. It would be interesting to find out more about the properties of such integrals.

Now, we make some notes about spaces we will deal with. Complete bornological locally convex spaces include: (1) all Banach spaces, in general nonseparable. The decomposition of these into inductive limits of separable Banach spaces gives a new point of view on the whole $L(\mathbf{X}, \mathbf{Y})$-measure theory in Banach spaces. (2) all Fréchet spaces (= the complete metrizable linear spaces). (3) a large number of non-metrizable locally convex spaces: various types of nuclear spaces, Schwartz spaces, DF-spaces and LB-spaces, etc.,cf. the book by S.M.Khaleelulla, [15], many of which have their origin in practical needs of theoretical physics.

1. Lattice structure in complete bornological locally convex spaces

1.1. Definition.
(a) Denote by $2^{\mathbf{X}}$ the set of all subsets of the set $\mathbf{X} \neq \emptyset$. Let $\mathcal{B}_{\mathbf{X}} \subset 2^{\mathbf{X}}$, be an ideal of the $\cup$- semilattice $2^{\mathbf{X}}$, cf. [1], II.3, with the additional proper- ty that every singleton belongs to $\mathcal{B}_{\mathbf{X}}$, i.e.

(i) $a \in \mathcal{B}_{\mathbf{X}}, b \in \mathcal{B}_{\mathbf{X}} \Rightarrow a \cup b \in \mathcal{B}_{\mathbf{X}}$,
(ii) $a \subset b, b \in \mathcal{B}_{\mathbf{X}} \Rightarrow a \in \mathcal{B}_{\mathbf{X}}$,
(iii) $\mathbf{x} \in \mathbf{X} \Rightarrow \{\mathbf{x}\} \in \mathcal{B}_{\mathbf{X}}$.

Then we say that the system $\mathcal{B}_{\mathbf{X}}$ of sets defines a bornology on $\mathbf{X}$, we denote it by $\mathbf{B}_{\mathbf{X}}$, cf. [18], chap. 2., §1, Definition 1.
(b) If $a \in \mathcal{B}_{\mathbf{X}}$, then we will also say that the set a is $\mathbf{B}_{\mathbf{X}}$- bounded.
(c) We say that the set $\mathcal{U} \subset \mathcal{B}_{\mathbf{X}}$ is a basis of the bornology $\mathbf{B}_{\mathbf{X}}$ if for every $b \in \mathcal{B}_{\mathbf{X}}$ there is a set $u \in \mathcal{U}$, such that $b \subset u$.

1.2. Example. It is easy to verify that the set $\mathcal{B}_{\mathbf{C}}$ of all sets of the first category on the real line $\mathbb{R}$ forms a bornology, $\mathbf{B}_{\mathbf{C}}$. Denote by $\mathbf{B}_{\mathbf{N}}$ the (von Neumann) bornology of sets bounded in the classical sense on $\mathbb{R}$. Then there are sets which (a) are of the first category but are not bounded, (b) are bounded but are not of the first category. So, $\mathcal{B}_{\mathbf{C}} \not\subset \mathcal{B}_{\mathbf{N}}$ and $\mathcal{B}_{\mathbf{N}} \not\subset \mathcal{B}_{\mathbf{C}}$, where $\mathcal{B}_{\mathbf{C}}, \mathcal{B}_{\mathbf{N}}$ are the set systems defining the bornologies $\mathbf{B}_{\mathbf{C}}, \mathbf{B}_{\mathbf{N}}$, respectively. Clearly $\mathcal{B}_{\mathbf{C}} \cap \mathcal{B}_{\mathbf{N}}$ defines a bornology, $\mathbf{B}_{\mathbf{C} \cap \mathbf{N}}$. $\square$

1.3. Definition. Let $\mathbf{X}$ be a Hausdorff topological vector space over the field $\mathbb{K}$ of real $\mathbb{R}$ or complex $\mathbb{C}$ numbers equipped with the bornology $\mathbf{B}_{\mathbf{X}}$.

A (separable) Banach disk in the space $\mathbf{X}$ is a set $u \in \mathbf{B}_{\mathbf{X}}$ which is clos- ed, absolutely convex and the linear span $\mathbf{X}_u$ of which is a (separable) Banach space.

1.4. Definition. Let $\mathbf{X}$ be a Hausdorff topological vector space over the field $\mathbb{K}$ equipped with the bornology $\mathbf{B}_{\mathbf{X}}$ with basis $\mathcal{U}$ of all separable Banach disks. Let on $\mathcal{U}$ an order be given by inclusion of sets. If $\mathbf{X}$ is an inductive limit of Banach spaces $\mathbf{X}_u, u \in \mathcal{U}$, i.e.

$$\mathbf{X} = \mathrm{ind}_{u \in \mathcal{U}} \mathbf{X}_u,$$

then we say that $\mathbf{X}$ is a complete bornological locally convex space, cf. [18], chap. 4., § 4, Th.1.

For the fact that the Banach spaces in the inductive limit in the definition of the complete bornological locally convex spaces can be chosen to be separable see [13], 13.2, Th. 3.

1.5. Remark. There are given three structures on $\mathbf{X}$: vector, topological and bornological. These structures are pairwise compatible, i.e.

(a) vector operations are continuous,

(b.i) $a \in \mathcal{B}_{\mathbf{X}}, b \in \mathcal{B}_{\mathbf{X}} \Rightarrow a + b \in \mathcal{B}_{\mathbf{X}}$,
where $a + b = \{\mathbf{x} = \mathbf{x}_a + \mathbf{x}_b; \mathbf{x}_a \in a, \mathbf{x}_b \in b\}$,

(b.ii) $a \in \mathcal{B}_{\mathbf{X}} \Rightarrow Sa \in \mathcal{B}_{\mathbf{X}}$, where $S \subset \{k \in \mathbb{K}; |k| \leq 1\}$,

(c.i) the bornology $\mathbf{B}_{\mathbf{X}}$ is stronger than von Neumann bornology $\mathbf{B}_{\mathbf{N}}$ on $\mathbf{X}$, i.e. $\mathcal{B}_{\mathbf{X}} \subset \mathcal{B}_{\mathbf{N}}$, where
$\mathcal{B}_{\mathbf{N}} = \{a \in 2^{\mathbf{X}}; a \text{ is absorbed by every zero neighborhood}\}$.

(c.ii) the topological closure of a $\mathbf{B}_{\mathbf{X}}$-bounded set is a $\mathbf{B}_{\mathbf{X}}$-bounded set.

1.6. Example. Let $\mathbf{X} = \mathcal{D}$ be the space of all infinitely many times differentiable real functions with compact supports on the real line. It is well-known, cf. [19], that

$$\mathcal{D} = \mathrm{ind}_{n \to \infty} \mathcal{D}_{[-n,+n]},$$

where

$$\mathcal{D}_{[-n,+n]} = \mathrm{ind}_{m \to \infty} \mathcal{D}^m_{[-n,+n]}, n \in \mathbb{N},$$

are Fréchet spaces for every positive integer $n \in \mathbb{N}$ and $\mathcal{D}^m_{[-n,+n]}$, $n, m \in \mathbb{N}$, is a Banach space equipped with the norm

$$p_{n,m}(\mathbf{x}) = \sup_{\substack{\xi \in [-n,+n] \\ 0 \leq k \leq m}} |\mathbf{x}^{(k)}(\xi)|, \tag{1}$$

where $\mathbf{x}^{(k)}(\xi)$ denotes the k-th derivative of the function $\mathbf{x} \in \mathcal{D}^m_{[-n,+n]}$ at the point $\xi \in [-n, +n]$, $0 \leq k \leq m$. The space $\mathcal{D}$ is a (von Neumann) bornological locally convex topological non-metrizable vector space, see e.g. [19], Appendix 2. It is easy to prove the following assertion: $a \subset \mathcal{D}$ is $\mathbf{B}_{\mathbf{N}}$-bounded iff there exists $n \in \mathbb{N}$, such that $a \subset \mathcal{D}_{[-n,+n]}$ and a is (von Neumann) bounded in the space $\mathcal{D}_{[-n,+n]}$, i.e. $\forall \mathbf{x} \in a$, $\forall m \in \mathbb{N}$, $\exists c_{n,m} \in \mathbb{N}$: $p_{n,m}(\mathbf{x}) \leq c_{n,m} < \infty$.

Denote by $\mathcal{U}$ the set of all $u \subset \mathcal{D}$, such that

$$u = \left\{ \sum_{i=1}^{I} \alpha_i \mathbf{e}_i; \sum_{i=1}^{I} |\alpha_i| \leq 1, \mathbf{e}_i \in \mathcal{D}, \alpha_i \in \mathbb{R} \right\}, \tag{2}$$

where $\mathbf{e}_i \in \mathcal{D}$ are linearly independent functions over $\mathbb{R}$, $i = 1, 2, \ldots, I$.

The finite number of points $\mathbf{e}_i \in \mathcal{D}$, $i = 1, 2, \ldots, I$, is a $\mathbf{B}_{\mathbf{N}}$-bounded set, and the absolute convex envelope of the set $E = \{\mathbf{e}_i \in \mathcal{D}; i = 1, 2, \ldots, I\}$ is again a $\mathbf{B}_{\mathbf{N}}$-bounded set. From the finiteness of the set $E \subset \mathcal{D}$ it follows that the set $u \in \mathcal{U}$ is closed in $\mathcal{D}$. Denote by p_u the Minkowski functional of the set u and $\mathbf{X}_u$ the linear envelope of the set $u \in \mathcal{U}$. We have a good criterion how to find out whether the function $\mathbf{f} \in \mathcal{D}$ belongs to $\mathbf{X}_u$ or not. If the Wronskian $W(\xi) = 0$ for every $\xi \in (-\infty, +\infty)$, where

$$W(\xi) = \det \begin{pmatrix} \mathbf{e}_1(\xi), & \mathbf{e}_2(\xi), & \ldots & \mathbf{e}_I(\xi), & \mathbf{f}(\xi) \\ \mathbf{e}'_1(\xi), & \mathbf{e}'_2(\xi), & \ldots & \mathbf{e}'_I(\xi), & \mathbf{f}'(\xi) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathbf{e}^{(I)}_1(\xi), & \mathbf{e}^{(I)}_2(\xi), & \ldots & \mathbf{e}^{(I)}_I(\xi), & \mathbf{f}^{(I)}(\xi) \end{pmatrix},$$

then $\mathbf{f} \in \mathbf{X}_u$.

It is easy to show that $\mathbf{X}_u$ is a Banach space equipped with the norm p_u. From the construction of the set $u \in \mathcal{U}$ it follows that the convergence with respect to the norm p_u in $\mathbf{X}_u$ is implied by the convergence in the Fréchet space $\mathcal{D}_{[-k,+k]}$ given by the countable system of norms
$p_m = p_{k,m}, m \in \mathbb{N}$, where $k = \max_{i=1,2,\dots,I}\{n_i; \mathbf{e}_i \in \mathcal{D}_{[-n_i,+n_i]}\}$.

It is easy to see that the family of all sets $u \in \mathcal{U}$ forms a Banach disk basis of a bornology on $\mathbf{X}$, denote it by $\mathbf{B}_1$. The bornology $\mathbf{B}_1$ is very poor: the $\mathbf{B}_1$--bounded sets are sets of $\mathbf{B}_{\mathbf{N}}$-bounded linear combinations of the sets from the discrete bornology. The inductive limit

$$\mathbf{X} = \mathrm{ind}_{u \in \mathcal{U}} \mathbf{X}_u$$

is an example of a set $\mathcal{D}$ equipped with the inductive convergence which differs from the convergence given by the system of seminorms $p_{n,m},\ n, m \in \mathbb{N}$, see (1). □

1.7. Lemma. Let $\mathcal{U}$ be a (separable) Banach disk basis of the bornol- ogy $\mathbf{B}_{\mathbf{X}}$. Let $u_1, u_2, u_3 \in \mathcal{U}$. Put

(i) $u_1 \wedge u_2 = u_1 \cap u_2$,
(ii) $u_1 \vee u_2 = \mathrm{acs}(u_1 \cup u_2)$,

where acs denotes the topological closure of the absolutely convex span of the set. Then

(a) $u_1 \wedge u_2 \in \mathcal{U}$,
(b) $u_1 \vee u_2 \in \mathcal{U}$,
(c) $u_1 \vee (u_2 \wedge u_3) = (u_1 \vee u_2) \wedge (u_1 \vee u_3)$.

Proof. (a) The intersection of two convex (closed, circled) sets is a convex (closed, circled) set. Show that $u_1 \wedge u_2$ generates a Banach space.

Let $\mathbf{x} \in \mathbf{X}_{u_1} \cap \mathbf{X}_{u_2}$. Then there exist $\lambda_1, \lambda_2 > 0$, such that $\mathbf{x} \in \lambda_1 u_1$ and $\mathbf{x} \in \lambda_2 u_2$. Then $\mathbf{x} \in \lambda u_1$ and $\mathbf{x} \in \lambda u_2$, where $\lambda = \max\{\lambda_1, \lambda_2\}$. This implies $\mathbf{x} \in \lambda(u_1 \cap u_2)$, i.e. $\mathbf{X}_{u_1} \cap \mathbf{X}_{u_2} \subset \mathbf{X}_{u_1 \cap u_2}$. The inverse inclusion is trivial. So, $\mathbf{X}_{u_1} \cap \mathbf{X}_{u_2} = \mathbf{X}_{u_1 \cap u_2}$. The assertion follows now from the fact that the intersection of two Banach spaces $(\mathbf{X}_{u_1}, p_{u_1})$ and $(\mathbf{X}_{u_2}, p_{u_2})$ is a Banach space with the norm $p(\mathbf{x}) = p_{u_1}(\mathbf{x}) + p_{u_2}(\mathbf{x})$, $\mathbf{x} \in \mathbf{X}_{u_1} \cap \mathbf{X}_{u_2}$.

(b) We have only to show that the set $u_1 \vee u_2$ is $\mathbf{B}_{\mathbf{X}}$-bounded. The other needed properties are satisfied. By Definition 1.1.(i) clearly $u_1 \cup u_2 \in \mathcal{B}_{\mathbf{X}}$. The absolutely convex envelope of the set $u_1 \cup u_2$ consists of elements $\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2$, where $|\lambda_1| + |\lambda_2| \leq 1$ and $\mathbf{x}_1, \mathbf{x}_2 \in u = u_1 \cup u_2$. We have

$$\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 \in u + u \in \mathcal{B}_{\mathbf{X}}$$

by the compatibility of the vector structure and the bornology, Remark 1.5.(b.i). Now we apply the fact that the topological and bornological structures are compatible, Remark 1.5.(c.ii).

(c) Since $\mathrm{acs}(u_1 \cap u_2) = \mathrm{acs}(u_1) \cap \mathrm{acs}(u_2)$, we have: $u_1 \vee (u_2 \wedge u_3) = \mathrm{acs}[u_1 \cup (u_2 \cap u_3)] = \mathrm{acs}[(u_1 \cup u_2) \cap (u_1 \cup u_3)] = \mathrm{acs}(u_1 \cup u_2) \cap \mathrm{acs}(u_1 \cup u_3) = (u_1 \vee u_2) \wedge (u_1 \vee u_3)$. □

1.8. Remark. In what follows Lemma 1.7. implies that we can suppose that the considered bases of bornologies are lattices. This fact implies again that the majority of considered objects in our theory will be (two - sided) nets on the lattices because in these objects the bases of bornologies will play also the role of the index sets.

1.9. Definition. We say that two bases $\mathcal{U}, \mathcal{V}$ of two bornologies $\mathbf{B}_{\mathcal{U}}, \mathbf{B}_{\mathcal{V}}$ on $\mathbf{X}$ are (bornologically) equivalent if $\mathbf{B}_{\mathcal{U}} = \mathbf{B}_{\mathcal{V}}$.

1.10. Lemma. Let $\mathbf{X}$ be a complete bornological locally convex space with (separable) Banach disk basis $\mathcal{V}$ of the bornology $\mathbf{B}_{\mathbf{X}}$. Let u_1 be an arbitrary element of $\mathcal{V}$, such that $u_1 \neq \{0\}$. Then we can choose a basis $\mathcal{U}$ equivalent to the basis $\mathcal{V}$, that consists of (separable) Banach disks such that $q \supset u_1$ for every $q \in \mathcal{U}$.

Proof. Let $\mathcal{V}$ be a Banach disk basis of the bornology $\mathbf{B}_{\mathbf{X}}$. Since $u_1, q \in \mathcal{V}$ are $\mathbf{B}_{\mathbf{X}}$-bounded, $u_1 + q$ is $\mathbf{B}_{\mathbf{X}}$-bounded, too. Since $\mathcal{V}$ is a basis, there exists $p \in \mathcal{V}$, such that $u_1 + q \subset p$. Take the system

$$\mathcal{U} = \{p \in \mathcal{V}; \exists q \in \mathcal{V}, u_1 + q \subset p\}.$$

We show that $\mathcal{U}$ generates the bornology $\mathbf{B}_X$. To prove this it is enough to show that $\mathcal{V} \leq \mathcal{U} \leq \mathcal{V}$.

$\mathcal{V} \leq \mathcal{U}$ (i.e.$\forall p \in \mathcal{U}, \exists v \in \mathcal{V} : p \subset v$), there holds trivially putting $v = p$, because $\mathcal{U} \subset \mathcal{V}$.

Show that $\mathcal{U} \leq \mathcal{V}$ (i.e. $\forall v \in \mathcal{V}, \exists p \in \mathcal{U} : v \subset p$). We have: $v = v + \{0\} \subset v + u_1 \subset p$. $\square$

1.11. Definition. We say that the basis $\mathcal{U}$ of the bornology $\mathbf{B}_{\mathbf{X}}$ on $\mathbf{X}$ has a marked element $u_1 \in \mathcal{U}, u_1 \neq \{0\}$, if the following property holds:

$$q \in \mathcal{U} \Rightarrow q \supset u_1.$$

1.12. Lemma. Let $\mathcal{U}$ be a basis of the bornology $\mathbf{B}_{\mathbf{X}}$ and $u_1 \in \mathcal{U}, u_1 \neq \{0\}$. Then the basis $\mathcal{U}_1$, where $\mathcal{U}_1 = \{u \in \mathcal{U}; u_1 \leq u\}$, of the bornology $\mathbf{B}_{\mathbf{X}}$ with the marked element u_1 is a Boolean ring with smallest element u_1.

Proof. Let $u_1 \leq u_2 \leq u_3, u_1, u_2, u_3 \in \mathcal{U}$. Put $z = (u_3 \setminus u_2) \vee u_1$. We have to prove: $z \vee u_2 = u_3, z \wedge u_2 = u_1$. Indeed, $z \vee u_2 = [(u_3 \setminus u_2) \vee u_1] \vee u_2 = (u_3 \setminus u_2) \vee (u_1 \vee u_2) = (u_3 \setminus u_2) \vee u_2 = \mathrm{acs}(u_3) = u_3$. Since $\mathcal{U}$ is a distributive lattice, we have: $z \wedge u_2 = [(u_3 \setminus u_2) \vee u_1] \wedge u_2 = [(u_3 \setminus u_2) \wedge u_2] \vee [u_1 \wedge u_2] = \emptyset \vee u_1 = u_1$. $\square$

1.13. Example. Continue Example 1.6. Note that $u_1 \wedge u_2$ may be $\{0\}$ when compacts defining Banach disks $u_1, u_2 \in \mathcal{U}$ are disjoint. Then there is only a trivial intersection element, the identical zero function.

Let $u_K, u_H \in \mathcal{U}$ be two Banach disks of the form (2) with compact supports $K, H \subset \mathbb{R}$ of the function spaces $\mathbf{X}_{u_K} \subset \mathcal{D}, \mathbf{X}_{u_H} \subset \mathcal{D}$, respectively. Then clearly $u_K \subset u_{H \cup K}, u_H \subset u_{H \cup K}$, and $u_H + u_K \subset 2u_{H \cup K} \in \mathcal{U}$. Using the notation $u_1 = u_H$ in the previous lemma, we may put $p = 2u_{H \cup K}$ and hence $\mathcal{U} = \{2u_{H \cup K} \in \mathcal{V}; K \subset \mathbb{R}$ is a compact$\}$. $\square$

1.14. Definition. Let $\mathbf{X}$ be a vector space equipped with the bornology $\mathbf{B}_{\mathbf{X}}$ given by the system of sets $\mathcal{U} \subset 2^{\mathbf{X}}$ (i.e. the basis of the bornology). We say that the net $\mathbf{x}_\omega \in \mathbf{X}, \omega \in \Omega$, is $\mathcal{U}$- convergent to $0 \in \mathbf{X}$ if there exists a set $u \in \mathcal{U}$, such that $0 \in u$ and for every $\delta > 0$ there exists $\omega_1 = \omega_1(\delta) \in \Omega$, such that for every $\omega \geq \omega_1, \omega \in \Omega$, there holds $\mathbf{x}_\omega \in \delta u$. We say that the net $\mathbf{x}_\omega \in \mathbf{X}, \omega \in \Omega$, is $\mathcal{U}$- convergent to the element $\mathbf{x} \in \mathbf{X}$ if the net $(\mathbf{x} - \mathbf{x}_\omega), \omega \in \Omega$, is $\mathcal{U}$- convergent to 0. To be more precise, we will sometimes call this the u- convergence of nets of elements from $\mathbf{X}$ to show explicitly which $u \in \mathcal{U}$ we have in the mind.

1.15. Lemma. If $\mathcal{U}, \mathcal{V}$ are two equivalent bases of a bornology $\mathbf{B_X}$ on $\mathbf{X}$, and $\mathbf{x}_\omega \in \mathbf{X}, \omega \in \Omega$, is a net $\mathcal{U}$- converging to the point $\mathbf{x} \in \mathbf{X}$, then the net $\mathbf{x}_\omega, \omega \in \Omega$, also $\mathcal{V}$- converges to $\mathbf{x}$.

Proof. Let $\delta > 0$ be given. By assumption there exists an element $u \in \mathcal{U}$, such that $\{0\} \in u$ and there is $\omega_1 = \omega_1(\delta) \in \Omega$, such that for every $\omega \geq \omega_1, \omega \in \Omega$, there holds $(\mathbf{x}_\omega - \mathbf{x}) \in \delta u$. Since $\mathcal{V}$ is a basis of the bornology $\mathbf{B_X}$, too, and u is $\mathbf{B_X}$-bounded, there is $v \in \mathcal{V}$, such that $u \subset v$, i.e. also $\delta u \subset \delta v$. □

1.16. Lemma. Let $\mathbf{X}$ be a bornological vector space. Let $\mathbf{x}_\omega, \omega \in \Omega$, be a net in $\mathbf{X}$ and there exist $u_1, u_2 \in \mathcal{U}$, such that for every $\delta \geq 0$ there exist $\omega_1, \omega_2 \in \Omega$, such that for every $\omega \geq \omega_1, \omega \geq \omega_2$, there holds $(\mathbf{x}_\omega - \mathbf{x}_1) \in \delta u_1, (\mathbf{x}_\omega - \mathbf{x}_2) \in \delta u_2$, respectively. Then $\mathbf{x}_1 = \mathbf{x}_2$.

Proof. The assertion is a consequence of the definition of the direction Ω and of Definition 1.14. We have: $\mathbf{x}_\omega - \mathbf{x}_1 \in \delta u_1, \mathbf{x}_\omega - \mathbf{x}_2 \in \delta u_2$ imply $\mathbf{x}_1 - \mathbf{x}_2 \in \delta u_1 - \delta u_2 = \delta(u_1 - u_2) \subset \delta u$, where $u \in \mathcal{U}, u_1 - u_2 \subset u$. □

1.17. Remark. Reformulate the definition of the bornological convergence above by Minkowski functional p_u (If $u \in \mathcal{U}$ does not absorb $\mathbf{x} \in \mathbf{X}$, we put $p_u(\mathbf{x}) = \infty$. Further, we use the traditional convention in measure and integration theory: $0 \cdot \infty = 0$ to simplify trivial assertions.). So, the net $\mathbf{x}_\omega \in \mathbf{X}, \omega \in \Omega$, $\mathcal{U}$-converges to $\mathbf{x} \in \mathbf{X}$ if and only if there exists an element $u \in \mathcal{U}$, such that $\lim_{\omega \in \Omega} p_u(\mathbf{x}_\omega - \mathbf{x}) = 0$. Since $\mathbf{X}_u, u \in \mathcal{U}$, is a Banach space, in the case of complete bornological locally convex spaces it is enough to deal with sequences instead of nets, cf.[18], §5, also the relation of this convergence to convergences in the topological sense. In a metrizable locally convex space with von Neumann bornology, sequential convergence is equivalent to bornological convergence of sequences.

1.18. Example. The bornological convergence with respect to the von Neumann bornology in $\mathcal{D}$ is commonly known. What about the $\mathbf{B}_1$-convergence in Example 1.6.?

Let $\mathbf{x}_n \in \mathbf{X}, n \in \mathbb{N}$, be such that $\mathbf{x}_n \in \mathbf{X}_u$, $u \in \mathcal{U}$, see the formula (2). Then $\mathbf{x}_n = \sum_{i=1}^{I} \alpha_{i,n} \mathbf{e}_i$ u-converges to $\mathbf{x} \in \mathbf{X}_n$ iff $\exists \alpha_i \in \mathbb{R}, i = 1, 2, \dots, I$, such that $\lim_{n \to \infty} \alpha_{i,n} = \alpha_i$. In this case $\mathbf{x} = \sum_{i=1}^{I} \alpha_i \mathbf{e}_i$. It is easy to verify that this convergence implies the usual convergence of this sequence in $\mathcal{D}$, i.e. the convergence with respect to the metric given by the system of seminorms (1) in Fréchet space determined by the supports of functions $\mathbf{x}_n \in \mathcal{D}, n = 0, 1, 2, \dots$, where $\mathbf{x}_0 = \mathbf{x}$. □

2. Lattices of set functions

2.1. Definition. Let $\mathbf{X}, \mathbf{Y}$ be two Hausdorff complete bornological locally convex spaces over the field $\mathbb{K}$ of real $\mathbb{R}$ or complex $\mathbb{C}$numbers equipped with the bornologies $\mathbf{B_X}, \mathbf{B_Y}$ on $\mathbf{X}, \mathbf{Y}$ with the Banach disk bases $\mathcal{U}, \mathcal{W}$ with marked elements $u_2 \in \mathcal{U}, w_2 \in \mathcal{W}$, respectively. On $\mathcal{U} \times \mathcal{W}$ define the operations $\vee, \wedge$ as follows. Let $(u_2, w_2), (u_3, w_3) \in \mathcal{U} \times \mathcal{W}$, then

(a) $(u_2, w_2) \vee (u_3, w_3) = (u_2 \vee u_3, w_2 \wedge w_3)$,

(b) $(u_2, w_2) \wedge (u_3, w_3) = (u_2 \wedge u_3, w_2 \vee w_3)$.

On $\mathcal{U} \times \mathcal{W}$ define an order $\ll$ as follows: for $(u, q), (p, w) \in \mathcal{U} \times \mathcal{W}$, put $(u, q) \ll (p, w)$ if and only if $u \leq p$ in $\mathcal{U}$ and $w \leq q$ in $\mathcal{W}$. Denote by $(\mathcal{U} \times \mathcal{W}, \vee, \wedge, \ll)$ the product $\mathcal{U} \times \mathcal{W}$ equipped with the operations $\vee, \wedge$ and the order $\ll$.

2.2. Lemma. The system $(\mathcal{U}\times\mathcal{W}, \vee, \wedge, \ll)$ given in Definition 2.1. is a distributive lattice. If $(u_2, w_2) \ll (u_3, w_3), (u_2, w_2), (u_3, w_3) \in \mathcal{U}\times\mathcal{W}$, then
the set

$$\{(u, w) \in \mathcal{U}\times\mathcal{W}; (u_2, w_2) \ll (u, w) \ll (u_3, w_3)\}$$

is a Boolean algebra with (u_2, w_2) as null and (u_3, w_3) as unit.

Proof. Put $z = ((u_3 \setminus u) \vee u_2, (w_2 \setminus w) \vee w_3)$. It is easy to show, see Lemma 1.12., that $z \vee (u, w) = (u_3, w_3)$ and $z \wedge (u, w) = (u_2, w_2)$. $\square$

2.3. Lemma. Let $T \neq \emptyset$ be a set. Denote by $\mathcal{P} \subset 2^T$ a δ- ring of sets,
and by $\sigma(\mathcal{P})$ a σ- algebra of sets from T generated by $\mathcal{P}$. Then

$$G \in \sigma(\mathcal{P}), E \in \mathcal{P} \Rightarrow G \cap E \subset \mathcal{P}.$$

Proof. Let $G \in \sigma(\mathcal{P})$. Then there are pairwise disjoint sets $G_i \in \mathcal{P}$, such that $\bigcup_{i=1}^{\infty} G_i = G$. Let $E \in \mathcal{P}$. Then $G \cap E = E \setminus (E \setminus G) = E \setminus (E \setminus \bigcup_{i=1}^{\infty} G_i) = E \setminus \bigcap_{i=1}^{\infty}(E \setminus G_i) \in \mathcal{P}$. $\square$

2.4. Definition. Denote by χ_E the characteristic function of the set E and by $L(\mathbf{X}, \mathbf{Y})$ the space of all continuous linear operators $L : \mathbf{X} \to \mathbf{Y}$.

(a) A function $\mathbf{f} : T \to \mathbf{X}$ is called $\mathcal{P}$- simple if $\mathbf{f}(T)$ is a finite set and $\mathbf{f}^{-1}(\mathbf{x}) \in \mathcal{P}$ for every $\mathbf{x} \in \mathbf{X} \setminus \{0\}$.

The space of all $\mathcal{P}$- simple functions is denoted by $\mathcal{F}(\mathcal{P}, \mathbf{X})$.

(b) The integral of the function $\mathbf{f} \in \mathcal{F}(\mathcal{P}, \mathbf{X})$ on $E \in \sigma(\mathcal{P})$ with respect
to the charge $\mathbf{m} : \mathcal{P} \to L(\mathbf{X}, \mathbf{Y})$ (= the finitely additive vector measure) is
defined by

$$\int_E \mathbf{f}\ \mathrm{d}\mathbf{x} = \sum_{\mathbf{x}\in\mathbf{f}(T)\setminus\{0\}} \mathbf{m}(E \cap \mathbf{f}^{-1}(\mathbf{x}))\mathbf{x} = \int_T \mathbf{f}\chi_E\ \mathrm{d}\mathbf{m}.$$

2.5. Example. This example shows one characteristic construction of operator valued charges.

Let $(\mathcal{S}, \sum_{\mathcal{S}}, \nu), (\mathcal{V}, \sum_{\mathcal{V}}, \mu), (\mathcal{P}, \sum_{\mathcal{P}}, \varphi)$ be measure spaces with σ-finite non-negative measures. Denote by $(\mathcal{R}, \sum_{\mathcal{R}}, \lambda)$ the measure product of these spaces, where $\mathcal{R} = \mathcal{P}\times\mathcal{S}\times\mathcal{V}, \sum_{\mathcal{R}} = \sum_{\mathcal{P}} \otimes \sum_{\mathcal{S}} \otimes \sum_{\mathcal{V}}, \lambda = \varphi\otimes\nu\otimes\mu$. Let $\mathbf{X}, \mathbf{Y}$ be two real vector lattices of integrable functions on $(\mathcal{S}, \sum_{\mathcal{S}}, \nu), (\mathcal{V}, \sum_{\mathcal{V}}, \mu)$, respectively. Let $K : \mathcal{R} \to (-\infty, +\infty)$ be an integrable function on $(\mathcal{R}, \sum_{\mathcal{R}}, \lambda)$, cf [9]. The order on $\mathbf{X}, (\mathbf{Y})$ is defined as usual: $\mathbf{x}_1 \leq \mathbf{x}_2$ iff $\forall v \in \mathcal{V}$: $\mathbf{x}_1(v) \leq \mathbf{x}_2(v)$, (analogously for the lattice $\mathbf{Y}$). Denote by $\mathbf{X}^+$ the positive cone of the lattice $\mathbf{X}$.

Define the operator valued measure $\mathbf{m} : \mathcal{P} \to L(\mathbf{X}, \mathbf{Y})$ as an integral operator

$$[\mathbf{m}(E)\mathbf{x}](s) = \mathbf{y}_E(s) = \int_E \int_{\mathcal{V}} K(z, s, v)\mathbf{x}(v)\ \mathrm{d}\mu(v)\ \mathrm{d}\varphi(z), E \in \mathcal{P},$$

choosing the kernel K such that $\mathbf{m}(E)\mathbf{x} \in \mathbf{Y}$ for every $\mathbf{x} \in \mathbf{X}$. In particular, we can put $\mathbf{X} = \mathcal{D}$, cf. Example 1.6., $\mathcal{V} = (-\infty, +\infty)$, and μ the Lebesgue measure.

Note that $\mathbf{m}(E)\mathbf{x} \geq 0$ for every $\mathbf{x} \in \mathbf{X}^+$ iff $K(z, s, v) \geq 0$ λ-a.e., $(z, s, v) \in \mathcal{R}$. For the proof see [14], XI., p. 393. $\square$

2.6. Remark. Since $L \in L(\mathbf{X}, \mathbf{Y})$ is a continuous operator, every image $L(u), u \in \mathcal{U}$, is von Neumann bounded in $\mathbf{Y}$.

2.7. Definition. Let $\mathbf{m} : \mathcal{P} \to L(\mathbf{X}, \mathbf{Y})$ be a charge. Let $(u, w) \in \mathcal{U} \times \mathcal{W}$. For $\mathbf{f} \in \mathcal{F}(\mathcal{P}, \mathbf{X})$ put

$$\|\mathbf{f}\|_{E,u} = \sup_{t \in E} p_u(\mathbf{f}(t)), u \in \mathcal{U}.$$

For $E \in \sigma(\mathcal{P})$ define

$$\hat{\mathbf{m}}_{u,w}(E) = \sup p_w \left(\int_E \mathbf{f} \; \mathrm{d}\mathbf{m} \right), \tag{3}$$

where the supremum is taken over all $\mathbf{f} \in \mathcal{F}(\mathcal{P}, \mathbf{X})$ such that $\|\mathbf{f}\|_{E,u} \le 1$. The resulting set function $\hat{\mathbf{m}}_{u,w} : \sigma(\mathcal{P}) \to [0, +\infty]$ is said to be the u, w- semivariation of $\mathbf{m}$ and the family $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}} = \{\hat{\mathbf{m}}_{u,w}; (u, w) \in \mathcal{U} \times \mathcal{W}\}$ is said

to be the $\mathcal{U}, \mathcal{W}$- semivariation of $\mathbf{m}$.

2.8. Example. Let $\mathbf{X}, \mathbf{Y}$ be two Banach lattices with order unit norms. Let $\mathbf{m} : \mathcal{P} \to L(\mathbf{X}, \mathbf{Y})^+$ be a charge. Let $E \in \mathcal{P}$. Then there exists a monotone sequence of sets $A_N \in \mathcal{P}, A_N \subset E, N \in \mathbb{N}$, such that $E \setminus A_N \to \emptyset$ as $N \to \infty$, and $\hat{\mathbf{m}}(E) = \lim_{N\to\infty} \|\mathbf{m}(A_N)\|$.

Firstly we show that if $\mathbf{m} : \mathcal{P} \to L(\mathbf{X}, \mathbf{Y})^+$ is a charge, then $\mathbf{m}$ is monotone, i.e. $E \subset F,\ E, F \in \mathcal{P} \Rightarrow \mathbf{m}(E) \le \mathbf{m}(F)$.

Indeed, let $E, F \in \mathcal{P}, E \subset F$. Then $F \setminus E \in \mathcal{P}$ and $\mathbf{m}(F \setminus E) \in L(\mathbf{X}, \mathbf{Y})^+$. Since $L(\mathbf{X}, \mathbf{Y})$ is an ordered vector space, $\mathbf{m}(E) \le \mathbf{m}(E) + \mathbf{m}(F \setminus E) = \mathbf{m}(E)$.

Now, from the definition of the ordered vector spaces it follows that

$$\mathbf{x}_1 \le \mathbf{x}_2, \mathbf{x}_3 \le \mathbf{x}_4 \Rightarrow \mathbf{x}_1 + \mathbf{x}_3 \le \mathbf{x}_2 + \mathbf{x}_4, \tag{4}$$

for every $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \in \mathbf{X}$. Clearly $\|\mathbf{x}_1\| \le 1, \|\mathbf{x}_2\| \le 1 \Rightarrow \mathbf{x}_1, \mathbf{x}_2 \in [-\mathbf{u}, +\mathbf{u}]$, where $[\cdot, \cdot]$ denotes the order interval and $\mathbf{u} \in \mathbf{X}$ is an order unit of the Banach lattice $\mathbf{X}$. Let $E, F \in \mathcal{P}$ be disjoint sets. So, $-\mathbf{m}(E \cup F)\mathbf{u} = -\mathbf{m}(E)\mathbf{u} - \mathbf{m}(F)\mathbf{u} \le \mathbf{m}(E)\mathbf{x}_1 + \mathbf{m}(F)\mathbf{x}_2, +\mathbf{m}(E \cup F)\mathbf{u} = \mathbf{m}(E)\mathbf{u} + \mathbf{m}(F)\mathbf{u} \ge \mathbf{m}(E)\mathbf{x}_1 + \mathbf{m}(F)\mathbf{x}_2$. And therefore

$$\mathbf{m}(E)\mathbf{x}_1 + \mathbf{m}(F)\mathbf{x}_2 \in [-\mathbf{m}(E \cup F)\mathbf{u}, +\mathbf{m}(E \cup F)\mathbf{u}].$$

Since for each $E \in \sigma(\mathcal{P}), E_n \in \mathcal{P}, n \in \mathbb{N}$, the operator $\mathbf{m}(E \cap E_n)$ is positive, and $\mathbf{Y}$ is a Banach lattice satisfying (4), we have:

$$\begin{aligned} \hat{\mathbf{m}}(E) &= \sup_{\|\mathbf{x}_n\| \le 1, E_n \in \mathcal{P}, n=1,2,\dots,N} \left\| \sum_{n=1}^{N} \mathbf{m}(E \cap E_n)\mathbf{x}_n \right\|_{\mathbf{Y}} \\ &\le \sup_{E_n \in \mathcal{P}, n=1,2,\dots,N} \left\| \mathbf{m}(E \cap \bigcup_{n=1}^{N} E_n)\mathbf{u} \right\|_{\mathbf{Y}} \\ &= \lim_{N\to\infty} \|\mathbf{m}(A_N)\mathbf{u}\|_{\mathbf{Y}} \\ &= \lim_{N\to\infty} \|\mathbf{m}(A_N)\|_{L(\mathbf{X},\mathbf{Y})}, \end{aligned} \tag{5}$$

where $E_n \in \mathcal{P}, E_{n'} \cap E_{n''} = \emptyset,\ n', n'' = 1, 2, \dots, N$, and $A_N = \bigcup_{n=1}^{N} E_n \in \mathcal{P}, N \in \mathbb{N}$. It is easy to choose a sequence of sets $A_N, N \in \mathbb{N}$, such that $E \setminus A_N \to \emptyset$ as $N \to \infty$. The inverse inequality to (5) follows from (3). $\square$

2.9. Remark. Observe that Lemma 1.12. implies that $u \ne \{0\}$ and $w \ne \{0\}$. Since applying a finite number of lattice operations to elements in $\mathcal{U}$, or $\mathcal{W}$, we cannot obtain $\mathbf{X}$, or $\mathbf{Y}$, the cases $u = \mathbf{X}$, or $w = \mathbf{Y}$, cannot occur.

2.10. Lemma. Let $u \in \mathcal{U}, w \in \mathcal{W}$. Then $\hat{\mathbf{m}}_{u,w}(\emptyset) = 0$ and the u,w- semivariation $\hat{\mathbf{m}}_{u,w}, (u,w) \in \mathcal{U} \times \mathcal{W}$, is a monotone, subadditive, and nonne- gative set function on $\sigma(\mathcal{P})$.

Proof. Trivial. □

2.11.Lemma. Define the lattice operations on the set $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$ as follows:

(i) $\hat{\mathbf{m}}_{u_2,w_2} \wedge \hat{\mathbf{m}}_{u_3,w_3} = \hat{\mathbf{m}}_{u_2 \wedge u_3, w_2 \vee w_3}$,

(ii) $\hat{\mathbf{m}}_{u_2,w_2} \vee \hat{\mathbf{m}}_{u_3,w_3} = \hat{\mathbf{m}}_{u_2 \vee u_3, w_2 \wedge w_3}$,

for every $(u_2,w_2),(u_3,w_3) \in \mathcal{U} \times \mathcal{W}$. On $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$ define an order $\ll$ as follows:

for $(u,q),(p,w) \in \mathcal{U} \times \mathcal{W}$, put $\hat{\mathbf{m}}_{u,q} \ll \hat{\mathbf{m}}_{p,w}$ if and only if $(u,q) \ll (p,w)$ in $\mathcal{U} \times W$. Then

(a) the $\mathcal{U},\mathcal{W}$- semivariation of the charge $\mathbf{m}$ is a distributive lattice,

(b) if $(u_2,w_2) \ll (u_3,w_3), (u_2,w_2),(u_3,w_3),(u,w) \in \mathcal{U} \times \mathcal{W}$, then the set $\{\hat{\mathbf{m}}_{u,w}; \hat{\mathbf{m}}_{u_2,w_2} \ll \hat{\mathbf{m}}_{u,w} \ll \hat{\mathbf{m}}_{u_3,w_3}\}$ is a Bolean algebra with $\hat{\mathbf{m}}_{u_2,w_2}$ as null and $\hat{\mathbf{m}}_{u_3,w_3}$ as unit.

Proof. Trivial. □

2.12. Lemma. Let $E \in \sigma(\mathcal{P}), (u,w) \ll (p,q), (u,w),(p,q) \in \mathcal{U} \times \mathcal{W}$.
Then

$$\hat{\mathbf{m}}_{u,w}(E) \le \hat{\mathbf{m}}_{p,q}(E).$$

Proof. Trivial. □

2.13. Lemma. Denote by $\mathbf{X}', \mathbf{Y}'$ the topological duals of $\mathbf{X}, \mathbf{Y}$, respec- tively. For every $\mathbf{y}' \in \mathbf{Y}'$, the set function $\mathbf{y}'\mathbf{m} : \mathcal{P} \to \mathbf{X}'$ is an $\mathbf{X}'$- valued charge, where

$$\mathbf{y}'\mathbf{m}(E)\mathbf{x} = < \mathbf{m}(E)\mathbf{x}, \mathbf{y}' >, E \in \mathcal{P}.$$

Proof. Trivial. □

2.14. Definition. For every $\mathbf{y}' \in \mathbf{Y}', u \in \mathcal{U}, E \in \sigma(\mathcal{P})$, we define the u- variation of the charge $\mathbf{y}'\mathbf{m}$ by the equation

$$\mathbf{var}_u(\mathbf{y}'\mathbf{m}, E) = \sup \sum_{i=1}^{I} \left| (\mathbf{y}'\mathbf{m})(E \cap E_i)\mathbf{x}_i \right|,$$

where the supremum is taken over all finite, pairwise disjoint sets $E_i \in \mathcal{P}$, and over all finite sets of elements $\mathbf{x}_i \in u,\ i = 1,2,\dots,I$.

2.15. Lemma. Let $\mathbf{m}$ be a charge. Denote by $w^0 \in \mathbf{Y}'$ the absolute polar of the set $w \in \mathcal{W}$.

Then the u,w- semivariation of $\mathbf{m}$ can be expressed in the form

$$\hat{\mathbf{m}}_{u,w}(E) = \sup_{\mathbf{y}' \in w^0} \mathbf{var}_u(\mathbf{y}'\mathbf{m}, E), E \in \sigma(\mathcal{P}).$$

Proof. Analogously to the proof of Proposition 5 in [5], §4., p.55. □

2.16.Remark. Since $\mathbf{Y}$ is bornologically complete, in the definition of the bornological convergence with respect to the equibornology, cf. [18], p. 66, Proposition 2, it suffices to consider the sequences in the space $L(\mathbf{X}, \mathbf{Y})$. The basis of this equibornology consists of Banach disks. To be more precise,
$\mathcal{W}_{\mathcal{U}} = \{w_u \subset L(\mathbf{X}, \mathbf{Y}); L \in w_u \Leftrightarrow \|L\|_{u,w} = \sup_{\mathbf{x} \in u} p_w(L(\mathbf{x})) \le 1, (u,w) \in \mathcal{U} \times \mathcal{W}\}$.

2.17. Definition. Let $(u,w) \in \mathcal{U} \times \mathcal{W}$. By the u,w- variation of the charge $\mathbf{m}$, we mean a nonnegative set function $\mathbf{var}_{u,w}(\mathbf{m},\cdot) : \sigma(\mathcal{P}) \to [0,+\infty]$ defined by

$$\mathbf{var}_{u,w}(\mathbf{m},E) = \sup \sum_{i=1}^{I} \left\| \mathbf{m}(E \cap E_i) \right\|_{u,w}, E \in \sigma(\mathcal{P}),$$

where the supremum is taken over all finite pairwise disjoint sets $E_i \in \mathcal{P}$, $i = 1,2,\dots,I$. The family $\mathbf{var}_{\mathcal{U},\mathcal{W}}(\mathbf{m},\cdot) = \{\mathbf{var}_{u,w}(\mathbf{m},\cdot); (u,w) \in \mathcal{U} \times \mathcal{W}\}$ is said to be the $\mathcal{U},\mathcal{W}$- variation of $\mathbf{m}$.

2.18. Definition. Let $(u,w \in \mathcal{U} \times \mathcal{W})$. By the scalar u,w- semivariation of the charge $\mathbf{m}$ we mean a nonnegative set function $\|\mathbf{m}\|_{u,w} : \sigma(\mathcal{P}) \to [0,\infty]$ defined by

$$\|\mathbf{m}\|_{u,w}(E) = \sup \left\| \sum_{i=1}^{I} \lambda_i \mathbf{m}(E \cap E_i) \right\|_{u,w}, E \in \sigma(\mathcal{P}),$$

where the supremum is taken over all finite sets of scalars $\|\lambda_i\| \leq 1$, $i = 1,2,\dots,I$, and over all disjoint sets $E_i \in \mathcal{P}, i = 1,2,\dots,I$. The family $\|\mathbf{m}\|_{\mathcal{U},\mathcal{W}} = \{\|\mathbf{m}\|_{u,w}; (u,w) \in \mathcal{U} \times \mathcal{W}\}$ is said to be the scalar $\mathcal{U},\mathcal{W}$- semi- variation of $\mathbf{m}$.

2.19. Lemma. The assertions of Lemmas 2.10., 2.11., and 2.12. remain valid if the $\mathcal{U},\mathcal{W}$- variation, or the scalar $\mathcal{U},\mathcal{W}$- semivariation is substituted for the $\mathcal{U},\mathcal{W}$- semivariation of the charge $\mathbf{m}$.

Proof. Trivial. $\square$

2.20. Remark. In what follows we will consider only set systems based on the notion of $\mathcal{U},\mathcal{W}$-semivariation (the $\mathcal{U},\mathcal{W}$-variation and scalar $\mathcal{U},\mathcal{W}$-semivariation can be used, as well).

3. Lattices of set systems, null sets

3.1. Definition. We say that a set $E \in \sigma(\mathcal{P})$ is of finite $\mathcal{U},\mathcal{W}$- semi- variation if there exists $(u_2,w_2) \in \mathcal{U} \times \mathcal{W}$, such that $\hat{\mathbf{m}}_{u_2,w_2}(E) < \infty$ (and also $\hat{\mathbf{m}}_{u,w}(E) < \infty$ for $(u,w) \ll (u_2,w_2), (u,w) \in \mathcal{U} \times \mathcal{W}$). If $E = T$, then we simply say that the charge $\mathbf{m}$ is of finite $\mathcal{U},\mathcal{W}$- semivariation.

3.2. Lemma. For $(u,w) \in \mathcal{U} \times \mathcal{W}$ denote by $\mathcal{P}_{u,w} \subset \mathcal{P}$ the greatest δ- ring of sets $E \in \mathcal{P}$ such that $\hat{\mathbf{m}}_{u,w}(E) < \infty$. If $(u_2,w_2), (u_3,w_3) \in \mathcal{U} \times \mathcal{W}$, then

$$(u_3,w_3) \ll (u_2,w_2) \Rightarrow \mathcal{P}_{u_3,w_3} \supset \mathcal{P}_{u_2,w_2} \Rightarrow \sigma(\mathcal{P}_{u_3,w_3}) \supset \sigma(\mathcal{P}_{u_2,w_2}).$$

Proof. The assertion follows from Lemma 2.10. $\square$

3.3. Lemma. Define the operations $\vee, \wedge$ on the family

$$\mathcal{P}_{\mathcal{U},\mathcal{W}} = \{\mathcal{P}_{u,w} \subset \mathcal{P}; (u,w) \in \mathcal{U} \times \mathcal{W}\},$$

as follows: for $(u_2,w_2), (u_3,w_3) \in \mathcal{U} \times \mathcal{W}$, put

$$\mathcal{P}_{u_2,w_2} \wedge \mathcal{P}_{u_3,w_3} = \mathcal{P}_{u_2 \vee u_3, w_2 \wedge w_3},$$

$$\mathcal{P}_{u_2,w_2} \vee \mathcal{P}_{u_3,w_3} = \mathcal{P}_{u_2 \wedge u_3, w_2 \vee w_3}.$$

Then $\mathcal{P}_{\mathcal{U},\mathcal{W}}$ is a distributive lattice. If $(u_2, w_2) \ll (u_3, w_3), (u_2, w_2), (u_3, w_3), (u, w) \in \mathcal{U} \times \mathcal{W}$, then the set $\{\mathcal{P}_{u,w} \in \mathcal{P}_{\mathcal{U},\mathcal{W}}; (u_2, w_2) \ll (u, w) \ll (u_3, w_3)\}$ is a Boolean algebra with $\mathcal{P}_{u_2,w_2}$ as null and $\mathcal{P}_{u_3,w_3}$ as unit.

Proof. The assertion is a consequence of Lemma 2.11. and 3.2. □

3.4. Remark. Let R be a Boolean algebra and let $M \subseteq R$ be a subset of R, such that $R \wedge M \subseteq M \neq \emptyset$. Recall that a topology on R, such that the operations $\triangle, \cap$ are continuous and the basis of zero neighborhoods consists of sets from M, is called the monotone ring topology. The set $\mathcal{M}(R)$ of all monotone ring topologies on R is a complete lattice with the trivial topology as its minimal element and the discrete topology as its maximal element, cf.[21].

3.5. Definition. We say that a net $E_i \in \sigma(\mathcal{P}), i \in I$, converges in the $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$- ring bornology, or $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$- ring converges, to $E \in \sigma(\mathcal{P})$ if there exists
a couple $(u, w) \in \mathcal{U} \times \mathcal{W}$, such that $\lim_{i \in I} \hat{\mathbf{m}}_{u,w}(E_i \triangle E) = 0$.

3.6 Remark. Clearly, this notion generalizes the convergence induced by the monotone ring topology in the context of Banach spaces.

3.7. Lemma.
(a) The operations $\triangle, \cap$ are continuous in the $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$- ring bornology.
(b) If there are monotone ring topologies on $\sigma(\mathcal{P})$ given by $\hat{\mathbf{m}}_{u_2,w_2}$, $\hat{\mathbf{m}}_{u_3,w_3}, (u_2, w_2), (u_3, w_3) \in \mathcal{U} \times \mathcal{W}, (u_2, w_2) \ll (u_3, w_3)$, then also $\hat{\mathbf{m}}_{u,w}$, $(u, w) \in \mathcal{U} \times \mathcal{W}, (u_2, w_2) \ll (u, w) \ll (u_3, w_3)$, defines the monotone ring topology on $\sigma(\mathcal{P})$.

Proof. (a) Let $E_\omega \in \sigma(\mathcal{P}), \omega \in \Omega, F_\theta \in \sigma(\mathcal{P}), \theta \in \Theta$, be two directed systems of sets $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-converging to $E \in \sigma(\mathcal{P})$ and $F \in \sigma(\mathcal{P})$, respectively. I. e., there exist $(r, s), (p, q) \in \mathcal{U} \times \mathcal{W}$, such that for every $\delta > 0$ there exist $\omega_1 \in \Omega, \theta_2 \in \Theta$, such that for every $\omega \geq \omega_1, \theta \geq \theta_2, \ \omega \in \Omega, \theta \in \Theta$, we have $\hat{\mathbf{m}}_{r,s}(E \triangle E_\omega) < \delta$ and $\hat{\mathbf{m}}_{p,q}(F \triangle F_\theta) < \delta$.

Let $\omega \geq \omega_1, \theta \geq \theta_2$. Then the subadditivity and monotonicity of the function $\hat{\mathbf{m}}_{r \wedge p, s \vee q}$ imply:

$$\hat{\mathbf{m}}_{r \wedge p, s \vee q}((E_\omega \triangle F_\theta) \triangle (E \triangle F)) = \hat{\mathbf{m}}_{r \wedge p, s \vee q}((E_\omega \triangle E) \triangle (F_\theta \triangle F)) \leq$$

$$\leq \hat{\mathbf{m}}_{r \wedge p, s \vee q}((E_\omega \triangle E) \triangle (F_\theta \triangle F)) + \hat{\mathbf{m}}_{r \wedge p, s \vee q}((F_\theta \triangle F) \setminus (E_\omega \triangle E)) \leq$$

$$\leq \hat{\mathbf{m}}_{r,s}((E_\omega \triangle E) + \hat{\mathbf{m}}_{p,q}((F_\theta \triangle F) < 2\delta.$$

Analogously for the intersection we have:

$$\hat{\mathbf{m}}_{r \wedge p, s \vee q}((E_\omega \cap F_\theta) \triangle (E \cap F)) \leq \hat{\mathbf{m}}_{r \wedge p, s \vee q}((E_\omega \triangle E) \cap (F_\theta \triangle F)) \leq$$

$$\leq \hat{\mathbf{m}}_{r,s}((E_\omega \triangle E) + \hat{\mathbf{m}}_{p,q}((F_\theta \triangle F) < 2\delta.$$

(b) The second assertion follows from Lemma 3.2. and (a). □

3.8. Remark. The continuity of operators $\mathbf{m}(E) \in L(\mathbf{X}, \mathbf{Y}), E \in \mathcal{P}$, is clearly a necessary condition for the continuity of the functions $\hat{\mathbf{m}}_{u,w}$, $(u, w) \in \mathcal{U} \times \mathcal{W}$, however, not a sufficient one.

3.9. Definition. We say that $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$ is continuous (from above in $\emptyset$) on $\sigma(\mathcal{P})$ if every nondecreasing sequence $\mathcal{R} \in \sigma(\mathcal{P})$ with the void intersection $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$- ring converges to the $\emptyset$.

3.10. Remark. Clearly, the continuity $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$ from above in $\emptyset$ generalizes the notion of continuity of the semivariation of a measure in $\emptyset$ from above in the context of Banach spaces.

3.11. Definition. Let $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$ be a $\mathcal{U},\mathcal{W}$-semivariation on $\sigma(\mathcal{P})$. We say that the set $N \in \sigma(\mathcal{P})$ is $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$- *null* if there exists a non-trivial couple $(u,w) \in \mathcal{U} \times \mathcal{W}$, such that $\hat{\mathbf{m}}_{u,w}(N) = 0$. Denote by $\mathcal{N}(\hat{\mathbf{m}}_{u,w})$ the set of all $\hat{\mathbf{m}}_{u,w}$- null sets and by $\mathcal{N}(\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}})$ the family of all $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$- null sets.

3.12. Remark. Note that for $u = \{0\}, w = Y$, each set $E \in \sigma(\mathcal{P})$ is $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-null.

3.13. Lemma. *The family $\mathcal{N}(\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}})$ is an ideal of subsets of $\sigma(\mathcal{P})$.*

Proof. Let $E, F \in \sigma(\mathcal{P}), \hat{\mathbf{m}}_{u_2,w_2}(E) = 0, \hat{\mathbf{m}}_{u_3,w_3}(F) = 0, (u_2,w_2)$, $(u_3,w_3) \in \mathcal{U} \times \mathcal{W}$. Then from the properties of the $\mathcal{U},\mathcal{W}$-semivariation of the charge $\mathbf{m}$ we get:

$$\begin{aligned}\hat{\mathbf{m}}_{r,s}(E \cup F) \le& \hat{\mathbf{m}}_{r,s}(E) + \hat{\mathbf{m}}_{r,s}(F)\\ \le& \hat{\mathbf{m}}_{u_2,s}(E) + \hat{\mathbf{m}}_{u_3,s}(F)\\ \le& \hat{\mathbf{m}}_{u_2,w_2}(E) + \hat{\mathbf{m}}_{u_3,w_3}(F)\\ =&0,\end{aligned}$$

where $w_2 \vee w_3 = s \in \mathcal{W}, u_2 \wedge u_3 = r \in \mathcal{U}$. Thus, $E \subset F \subset \mathcal{N}(\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}})$.

The property $E \in \mathcal{N}(\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}), F \subset E, F \in \sigma(\mathcal{P}) \Rightarrow F \in \mathcal{N}(\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}})$ is trivial. Hence, $\mathcal{N}(\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}})$ is an ideal of sets in $\sigma(\mathcal{P})$. $\square$

3.14. Lemma. For $(u_2,w_2),(u_3,w_3) \in \mathcal{U} \times \mathcal{W}$, define the lattice operations as follows:

$$\mathcal{N}(\hat{\mathbf{m}}_{u_2,w_2}) \vee \mathcal{N}(\hat{\mathbf{m}}_{u_3,w_3}) = \mathcal{N}(\hat{\mathbf{m}}_{u_2 \wedge u_3, w_2 \vee w_3}),$$

$$\mathcal{N}(\hat{\mathbf{m}}_{u_2,w_2}) \wedge \mathcal{N}(\hat{\mathbf{m}}_{u_3,w_3}) = \mathcal{N}(\hat{\mathbf{m}}_{u_2 \vee u_3, w_2 \wedge w_3}).$$

The family $\{\mathcal{N}(\hat{\mathbf{m}}_{u,w}); (u,w) \in \mathcal{U} \times \mathcal{W}\}$ is a distributive lattice of ideals of $\hat{\mathbf{m}}_{u,w}$-null sets, $w \in \mathcal{W}, u \in \mathcal{U}$, (the σ- ideals in the case that $\hat{\mathbf{m}}_{u,w}$ is a σ- subadditive set function).

Proof. Trivial. $\square$

3.15. Remark. Now we are able to introduce the"almost everywhere" notions: we say that a given assertion, definition, convergence, etc., holds $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a. e. if it holds everywhere except in a set $E \in \mathcal{N}(\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}})$.

4. An integral in complete bornological locally convex spaces

4.1. Lemma. Let $(u,w) \in \mathcal{U} \times \mathcal{W}$. If $E \in \sigma(\mathcal{P}), \mathbf{f} \in \mathcal{F}(\mathcal{P},\mathbf{X})$, then

$$q_w\left(\int_E \mathbf{f}\ \mathrm{d}\mathbf{m}\right) \le \|\mathbf{f}\|_{E,u}\ .\ \hat{\mathbf{m}}_{u,w}(E). \tag{6}$$

Proof. Trivial. $\square$

4.2. Remark. It is technically convenient to extend the definition of $\hat{\mathbf{m}}_{u,w}, u \in \mathcal{U}, w \in \mathcal{W}$, to an arbitrary subset of T. We do this as follows: if W is an arbitrary subset of T, then we define

$$\hat{\mathbf{m}}^*_{u,w}(W) = \inf_{E \in \sigma(\mathcal{P}), W \subset E} \hat{\mathbf{m}}_{u,w}(E).$$

4.3. Definition. Let $(u,w) \in \mathcal{U} \times \mathcal{W}$. We say that a sequence of charges $(\nu_i)_{i \in \mathbb{N}}, \nu_i : \sigma(\mathcal{P}) \to \mathbf{Y},\ \ i \in \mathbb{N}$, is $\hat{\mathbf{m}}_{u,w}$- equicontinuous if for every $\varepsilon > 0$, there are $i_1 = i_1(u,w,\varepsilon) \in \mathbb{N}$, and $E = E(u,w,\varepsilon) \in \sigma(\mathcal{P}_{u,w})$, such that for every $i \geq i_1, i \in \mathbb{N}$, and $D \subset T \setminus E, D \in \sigma(\mathcal{P})$, we have: $p_w(\nu_i(D)) < \varepsilon$.

4.4. Definition. Let $(u,w) \in \mathcal{U} \times \mathcal{W}$. We say that a sequence of charges $(\nu_i)_{i \in \mathbb{N}}, \nu_i : \sigma(\mathcal{P}) \to \mathbf{Y}, i \in \mathbb{N}$, is uniformly $\hat{\mathbf{m}}_{u,w}$- absolutely continuous if for every $\varepsilon > 0$, there are $i_2 = i_2(u,w,\varepsilon) \in \mathbb{N}$, and $\eta = \eta(u,w,\varepsilon) > 0$, such that for every $i \geq i_2, i \in \mathbb{N}$, the following implication holds:

$$A \in \sigma(\mathcal{P}), \hat{\mathbf{m}}_{u,w}(A) < \eta \Rightarrow p_w(\nu_i(A)) < \varepsilon. \tag{7}$$

4.5. Definition. Let $(u,w) \in \mathcal{U} \times \mathcal{W}$. We say that a sequence of func- tions $(\mathbf{f}_i)_{i \in \mathbb{N}}, \mathbf{f}_i : T \to \mathbf{X}, i \in \mathbb{N}$, $\hat{\mathbf{m}}_{u,w}$- converges to the function $\mathbf{f} : T \to \mathbf{X}$, if for every $\delta > 0, \eta > 0$, there is $i_3 = i_3(u,w,\delta,\eta) \in \mathbb{N}$, such that for every $i \geq i_3, i \in \mathbb{N}$, the following implication holds:

$$\hat{\mathbf{m}}^*_{u,w}(\{t \in T; p_u(\mathbf{f}_i(t) - \mathbf{f}(t)) \geq \delta\}) < \eta.$$

4.6. Definition. Let $(u,w) \in \mathcal{U} \times \mathcal{W}$.

(I) We say that a function $\mathbf{f} : T \to \mathbf{X}$ is $\mathcal{P}_{u,w}$- measurable if it belongs to the closure of the space $\mathcal{F}(\mathcal{P}_{u,w}, \mathbf{X})$ with respect to the topology of the $\hat{\mathbf{m}}_{u,w}$-convergence. We say that a function $\mathbf{f} : T \to \mathbf{X}$ is $\mathcal{P}_{\mathcal{U},\mathcal{W}}$- measurable if there exists a couple $(u,w) \in \mathcal{U} \times \mathcal{W}$, such that $\mathbf{f}$ is $\mathcal{P}_{u,w}$- measurable.

(II) We say that a $\mathcal{P}_{u,w}$- measurable function $\mathbf{f} : T \to \mathbf{X}$ is $\mathcal{P}_{u,w}$-integrable over T, we write $\mathbf{f} \in \mathcal{F}_{u,w}$, if there exists a sequence $(\mathbf{f}_i)_{i \in \mathbb{N}}$ in $\mathcal{F}(\mathcal{P}_{u,w}, \mathbf{X})$ satisfying the following conditions:

(a) the sequence $(\mathbf{f}_i)_{i \in \mathbb{N}}$ $\hat{\mathbf{m}}_{u,w}$-converges to $\mathbf{f}$,

(b) the sequence $(\mathbf{m}_{\mathbf{f}_i})_{i \in \mathbb{N}}$ is uniformly $\hat{\mathbf{m}}_{u,w}$-absolutely continuous,

(c) the sequence $(\mathbf{m}_{\mathbf{f}_i})_{i \in \mathbb{N}}$ is $\hat{\mathbf{m}}_{u,w}$-equicontinuous.

If $E \in \sigma(\mathcal{P})$, then the limit (see Remark 4.7.)

$$\int_E \mathbf{f}\ \mathrm{d}\mathbf{m} = \lim_{i \in \mathbb{N}} \int_E \mathbf{f}_i\ \mathrm{d}\mathbf{m} \tag{8}$$

is called an indefinite integral $\mathbf{m}_{\mathbf{f}}$ at the set E, cf. [2].

We say that a function $\mathbf{f} : T \to \mathbf{X}$ is $\mathcal{P}_{\mathcal{U},\mathcal{W}}$-integrable if there exists a couple $(u,w) \in \mathcal{U} \times \mathcal{W}$ such that $\mathbf{f}$ is $\mathcal{P}_{u,w}$-integrable. We then write $\mathbf{f} \in \mathcal{F}_{\mathcal{U},\mathcal{W}}$.

4.7. Remark. It can be proved analogously to [2] that the value of the integral in (8) is independent of the sequence of simple functions $(\mathbf{f}_i)_{i \in \mathbb{N}}$ in this definition.

4.8. Definition. We say that a sequence $(\mathbf{f}_i)_{i\in\mathbb{N}}$ of $\mathcal{P}$- simple $[\mathcal{P}_{\mathcal{U},\mathcal{W}}$- integrable$]$ functions is fundamental (converges) in the mean if the sequ-

ence of charges $(\mathbf{m}_{\mathbf{f}_i})_{i\in\mathbb{N}}$ is $\mathcal{W}$- fundamental ($\mathcal{W}$- converges) in $\mathbf{Y}$ uniformly for every $E \in \sigma(\mathcal{P})$, where

$\mathbf{m}_{\mathbf{f}}(.) = \int_{.} \mathbf{f} \; \mathrm{d}\mathbf{m} : \sigma(\mathcal{P}) \to \mathbf{Y}, \mathbf{f} \in \mathcal{F}(\mathcal{P}, \mathbf{X}), [\mathcal{F}_{\mathcal{U},\mathcal{W}}]$.

4.9. Theorem (Vitali)**.** Let $(u, w) \in \mathcal{U} \times \mathcal{W}$. Let $(\mathbf{f}_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{F}_{u,w}$ and $\mathbf{f} : T \to \mathbf{X}$ be a function, such that the condition (a), (b), (c) in Definition 4.6. are satisfied.

Then $\mathbf{f} \in \mathcal{F}_{u,w}$ and the sequence $(\mathbf{f}_i)_{i\in\mathbb{N}}$ converges in the mean to $\mathbf{f}$.

Proof. First consider the case when $\mathbf{f}_i \in \mathcal{F}(\mathcal{P}_{u,w}, \mathbf{X}), i \in \mathbb{N}$. Then by Definition 4.6., $\mathbf{f} \in \mathcal{F}_{u,w}$. Since $\mathbf{Y}_w$ is complete, it suffices to show that the sequence $(\mathbf{f}_i)_{i\in\mathbb{N}}$ is fundamental in the mean, i.e. we have to prove that the integral (8) is well-defined.

Let $F \in \sigma(\mathcal{P}), E \in \mathcal{P}, w \in \mathcal{W}$. We have:

$$d = p_w\left(\int_F \mathbf{f}_i \; \mathrm{d}\mathbf{m} - \int_F \mathbf{f}_j \; \mathrm{d}\mathbf{m}\right) =$$

$$= p_w\left(\int_{F\cap(T\setminus E)} \mathbf{f}_i \; \mathrm{d}\mathbf{m} - \int_{F\cap(T\setminus E)} \mathbf{f}_j \; \mathrm{d}\mathbf{m} + \int_{F\cap E} (\mathbf{f}_i - \mathbf{f}_j) \; \mathrm{d}\mathbf{m}\right), \tag{9}$$

where $i, j \in \mathbb{N}$. Clearly $F \cap (T \setminus E) \subset T \setminus E$ and $F \cap (T \setminus E) \in \sigma(\mathcal{P})$.

Let ε be an arbitrary positive number. Choose $E \in \mathcal{P}_{u,w}, i_1 \in \mathbb{N}$, such as in Definition 4.3. Put $D = F \cap (T \setminus E)$. Then, by Definition 4.3., for every $i, j \geq i_1$ we obtain:

$$d < 2\varepsilon + p_w\left(\int_{F\cap E} (\mathbf{f}_i - \mathbf{f}_j) \; \mathrm{d}\mathbf{m}\right). \tag{10}$$

Further, (6) implies:

$$p_w\left(\int_{F\cap E} \mathbf{f} \; \mathrm{d}\mathbf{m}\right) \leq \|\mathbf{f}\|_{E\cap F,u} \; . \; \hat{\mathbf{m}}_{u,w}(E \cap F), \tag{11}$$

where $\mathbf{f} \in \mathcal{F}(\mathcal{P}_{u,w}, \mathbf{X})$. Since $\mathbf{f}_i, \mathbf{f}_j \in \mathcal{F}(\mathcal{P}_{u,w}, \mathbf{X})$, $i, j \in \mathbb{N}$, then $\mathbf{f}_i - \mathbf{f}_j \in \mathcal{F}(\mathcal{P}_{u,w}, \mathbf{X})$, and, by (11), for every $i, j \in \mathbb{N}$ we have:

$$p_w\left(\int_{F\cap E} (\mathbf{f}_i - \mathbf{f}_j) \; \mathrm{d}\mathbf{m}\right) \leq \|\mathbf{f}_i - \mathbf{f}_j\|_{E\cap F,u} \; . \; \hat{\mathbf{m}}_{u,w}(E \cap F). \tag{12}$$

Since the charge $\mathbf{m}$ is of a finite u, w-semivariation on $E \in \mathcal{P}_{u,w}$ and $\hat{\mathbf{m}}_{u,w}$ is a monotone set function, we have $\hat{\mathbf{m}}_{u,w}(E \cap F) < \infty$, too. Then for a given $\varepsilon > 0$ there is $\delta > 0$, such that the following implication is true:

$$\mathbf{f}_i, \mathbf{f}_j \in \mathcal{F}(\mathcal{P}_{u,w}, \mathbf{X}), \|\mathbf{f}_i - \mathbf{f}_j\|_{E\cap F,u} < \delta \Rightarrow p_w\left(\int_{F\cap E} (\mathbf{f}_i - \mathbf{f}_j) \; \mathrm{d}\mathbf{m}\right) < \varepsilon. \tag{13}$$

Denote by $G = \{t \in F \cap E; p_u(\mathbf{f}_i(t) - \mathbf{f}_j(t)) < \delta\}$. Since $\mathbf{f}_i, \mathbf{f}_j \in \mathcal{F}(\mathcal{P}_{u,w}, \mathbf{X})$, $i, j \in \mathbb{N}$, there is $G \in \sigma(\mathcal{P})$. We have:

$$p_w\left(\int_{F\cap E} (\mathbf{f}_i - \mathbf{f}_j) \; \mathrm{d}\mathbf{m}\right) \leq$$

$$\leq p_w\left(\int_{(F\cap E)\cap G}(\mathbf{f}_i-\mathbf{f}_j)\ \mathrm{d}\mathbf{m}\right)+p_w\left(\int_{(F\cap E)\setminus G}(\mathbf{f}_i-\mathbf{f}_j)\ \mathrm{d}\mathbf{m}\right). \tag{14}$$

So, from (10), (13), and (14) we obtain:

$$d<3\varepsilon+p_w\left(\int_{(F\cap E)\setminus G}(\mathbf{f}_i-\mathbf{f}_j)\ \mathrm{d}\mathbf{m}\right). \tag{15}$$

By (b) the sequence $(\mathbf{m}_{\mathbf{f}_i})_{\mathbf{f}_i\in\mathbb{N}}$ is uniformly $\hat{\mathbf{m}}_{u,w}$-absolutely continuous, see Definition 4.4. Choose $i_2\geq i_1$. Further, if $p_w(\mathbf{m}_{\mathbf{f}_i})(A)<\varepsilon, A\in\sigma(\mathcal{P}),\ i,j\in\mathbb{N},\ i,j\geq i_2$, then

$$p_w(\mathbf{m}_{\mathbf{f}_i-\mathbf{f}_j})(A)<2\varepsilon. \tag{16}$$

By (a) the sequence $(\mathbf{f}_i)_{i\in\mathbb{N}}$ $\hat{\mathbf{m}}_{u,w}$-converges to $\mathbf{f}$, see Definition 4.5. Since $\hat{\mathbf{m}}_{u,w}$, is a monotone set function, then also the sequence $(\mathbf{f}_i\chi_A)_{i\in\mathbb{N}}$ $\hat{\mathbf{m}}_{u,w}$-converges to $\mathbf{f}\chi_A, A\in\sigma(\mathcal{P})$, i.e. for every $i\geq i_3,\ i,i_3\in\mathbb{N}$, we have

$$\hat{\mathbf{m}}_{u,w}(\{t\in A;p_u(\mathbf{f}_i(t)-\mathbf{f}_i(t))\geq\delta\})<\eta. \tag{17}$$

Choose $i_3\geq i_2$ and put $A=(F\cap E)\setminus G\in\sigma(\mathcal{P})$. Since $\mathbf{f}_i,\mathbf{f}_j\in\mathcal{F}(\mathcal{P}_{u,w},\mathbf{X}),\ i,j\in\mathbb{N}$, there is $\{t\in A;p_u(\mathbf{f}_i(t)-\mathbf{f}_j(t))\geq\delta\}\in\sigma(\mathcal{P})$. (In this case clearly $\hat{\mathbf{m}}^*_{u,w}=\hat{\mathbf{m}}_{u,w}$.) Then (7), (15), (16), and (17) imply that for every $F\in\sigma(\mathcal{P}),\varepsilon>0$, there is $i_3\in\mathbb{N}$, such that for every $i\geq i_3, i\in\mathbb{N}$, we have $d<5\varepsilon$.

Let us consider $\mathbf{f}_i\in\mathcal{F}_{u,w}, i\in\mathbb{N}$. By Definition 4.6. to every $\mathcal{P}_{u,w}$-integrable function there exists a sequence of functions $(\mathbf{f}_{i,j})_{j\in\mathbb{N}}$ in
$\mathcal{F}(\mathcal{P}_{u,w},\mathbf{X})$, such that the conditions of Definition 4.6., (a), (b), and (c) are satisfied.

It is easy to see that the diagonal subsequence $(\mathbf{f}_{i,i})_{i\in\mathbb{N}}$, of the sequence $(\mathbf{f}_{i,j})_{j\in\mathbb{N}}, (i,j)\in\mathbb{N}\times\mathbb{N}$, satisfies the conditions (a), (b), and (c) of Definition 4.6. □

4.10. Theorem (Lebesgue). Let $(u,w)\in\mathcal{U}\times\mathcal{W}$. Let $(\mathbf{f}_i)_{i\in\mathbb{N}}$ be a sequ- ence in $\mathcal{F}_{u,w},\mathbf{f}\colon T\to\mathbf{X}$ be a function, $E\in\sigma(\mathcal{P})$. Assume that

(a) the sequence $(\mathbf{f}_i)_{i\in\mathbb{N}}$ $\hat{\mathbf{m}}_{u,w}$- converges to the function $\mathbf{f}$,

(b) there is a function $\mathbf{g}\in\mathcal{F}_{u,w}$, such that the sequence $(\mathbf{m}_{\mathbf{f}_i})_{i\in\mathbb{N}}$ is such that

$$p_w\left(\int_E\mathbf{f}_i\ \mathrm{d}\mathbf{m}\right)\leq p_w\left(\int_E\mathbf{g}\ \mathrm{d}\mathbf{m}\right) \tag{18}$$

for every $i\in\mathbb{N}$.

Then $\mathbf{f}\in\mathcal{F}_{u,w}$ and the sequence $(\mathbf{f}_i)_{i\in\mathbb{N}}$ converges in the mean to $\mathbf{f}$.

Proof. This version of the Lebesgue Dominated Convergence Theorem immediately follows from Theorem 4.9. as a consequence. (Cf. also [6], [7], and [8].) □

4.11. Remark. Observe that the integral in Definition 4.6. was constructed under the assumption that the measure $\mathbf{m}$ is merely finitely additive. It is easy to see that the sum of two $\mathcal{P}_{\mathcal{U},\mathcal{W}}$-integrable functions need not be a $\mathcal{P}_{\mathcal{U},\mathcal{W}}$-integrable function. Nonlinear integrals, e.g. the Uryson integral operators are of this kind, cf. [16]. We show that under additional assumptions on the charge $\mathbf{m}$ we can define a linear integral in complete bornological locally convex spaces.

4.12. Definition. Denote by $\mathbf{m}_{u,w}(E)\mathbf{x} = \mathbf{m}(E)\mathbf{x}, E \in \mathcal{P}_{u,w}$, the restriction of a charge $\mathbf{m}$ to the set system $\mathcal{P}_{u,w}, (u,w) \in \mathcal{U} \times \mathcal{W}$. We say that a charge $\mathbf{m} : \mathcal{P} \to L(\mathbf{X}, \mathbf{Y})$ is an operator valued measure σ- additive in the equibornology of the space $L(\mathbf{X}, \mathbf{Y})$, shortly we say that it is $u - \mathcal{U}, \mathcal{W} - \sigma$-additive, if for every $(u,w) \in \mathcal{U} \times \mathcal{W}$ the set function $\mathbf{m}_{u,w}$ is a σ- additive vector measure in the uniform topology of the space $L(\mathbf{X}_u, \mathbf{Y}_w)$.

4.13. Remark. Clearly, Definition 4.12. generalizes the notion of a measure that is σ-additive in the uniform operator topology in the case when both $\mathbf{X}, \mathbf{Y}$ are Banach spaces.

4.14. Definition.

(a) Let $(u,w) \in \mathcal{U} \times \mathcal{W}$. We say that the restriction $\mathbf{m}_{u,w} : \mathcal{P}_{u,w} \to L(\mathbf{X}_u, \mathbf{Y}_w)$ of a $u - \mathcal{U}, \mathcal{W} - \sigma$- additive measure $\mathbf{m}$ has the *- property if there exists a non-negative finite σ- additive measure $\nu_{u,w} : \sigma(\mathcal{P}_{u,w}) \to [0, \infty)$, such that $\nu_{u,w}(E) \to 0$ if and only if $\hat{\mathbf{m}}_{u,w}(E) \to 0$, cf. [2], Definition 2.

(b) Let from the fact that both $\mathbf{m}_{u_2,w_2}, \mathbf{m}_{u_3,w_3}, (u_2, w_2), (u_3, w_3) \in \mathcal{U} \times \mathcal{W}$, have the *- property it follow that there is $(u_4, w_4) \in \mathcal{U} \times \mathcal{W}, u_2 \vee u_3 \subset u_4, w_2 \vee w_3 \subset w_4$, such that $\mathbf{m}_{u_4,w_4}$ has the *- property. Then we say that the $u - \mathcal{U}, \mathcal{W} - \sigma$- additive measure $\mathbf{m}$ has the GSB- property.

4.15. Example. Consider the measure $\mathbf{m}(E)\mathbf{x} = L(\mathbf{x})\lambda(E)$, where $L \in L(\mathbf{X}, \mathbf{Y}), \mathbf{x} \in \mathbf{X}, \lambda$ is the Lebesgue measure on the real line, $E \in S$ is a Lebesgue measurable set, and $L \in L(\mathbf{X}, \mathbf{Y})$ is a continuous linear operator. So, for every $u \in \mathcal{U}$ there exists an element $w \in \mathcal{W}$, such that $L(u) \subset w$. (In particular, if $\mathbf{X} = \mathbf{Y}, \mathcal{U} = \mathcal{W}$, and $L\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in \mathbf{X}$, then $\sigma \colon u \mapsto w = u$.) Take a couple $(u,w) \in \mathcal{U} \times \mathcal{W}$ and an arbitrary set $E \in \mathcal{P}$. Then from the definition of the u, w-semivariation we obtain:

$$\begin{aligned}
\hat{\mathbf{m}}_{u,w}(E) &= \sup_{\substack{\mathbf{x}_i \in u \\ j=1,2,\ldots,I}} p_w \left(\sum_{i=1}^{I} \mathbf{m}(E_i \cap E)\mathbf{x}_i \right) \\
&= \sup_{\substack{\mathbf{x}_i \in u \\ i=1,2,\ldots,I}} p_w \left(\sum_{i=1}^{I} \lambda(E_i \cap E) L(\mathbf{x}_i) \right) \\
&= \sup_{\substack{\mathbf{x}_i \in u \\ i=1,2,\ldots,I}} p_w \left(L \left\{ \sum_{i=1}^{I} \lambda(E_i \cap E)\mathbf{x}_i \right\} \right) \\
&= \sup_{\substack{\mathbf{x}_i \in u \\ i=1,2,\ldots,I}} p_w \left(L \left\{ \sum_{i=1}^{I} \frac{\lambda(E_i \cap E)}{\lambda(E)} \mathbf{x}_i \right\} \right) \cdot \lambda(E) \\
&\le \lambda(E) = \nu_{u,w}(E),
\end{aligned}$$

for every $(u,w) \in \mathcal{U} \times \mathcal{W}$, such that $L(u) \subset w$. The last inequality follows from the fact that $u \in \mathcal{U}$ is a convex set, and, therefore,

$$\sum_{i=1}^{I} \frac{\lambda(E_i \cap E)}{\lambda(E)} = 1, \mathbf{x}_i \in u, \ i = 1, 2, \ldots, I \Rightarrow \sum_{i=1}^{I} \frac{\lambda(E_i \cap E)}{\lambda(E)} \mathbf{x}_i \in u.$$

Clearly $\lambda(E) \to 0, L(u) \subset w \Rightarrow \hat{\mathbf{m}}_{u,w}(E) \to 0$. Conversely, fix an arbitrary $\mathbf{x} \in u, L(\mathbf{x}) \neq 0$. Then $\hat{\mathbf{m}}_{u,w}(E) \ge p_w(\mathbf{m}(E)\mathbf{x}) = p_w(\lambda(E)L(\mathbf{x})) = \lambda(E) \cdot p_w(L(\mathbf{x}))$.

Thus $\hat{\mathbf{m}}_{u,w}(E) \to 0 \Rightarrow \lambda(E) \to 0$. If $L(u_2) \subset w_2, L(u_3) \subset w_3,\ u_2, u_3 \in \mathcal{U},\ w_2, w_3 \in \mathcal{W}$, then put $u_4 = u_2 \vee u_3, w_4 = w_2 \vee w_3 \vee L(u_2 \vee u_3)$. We see that our measure $\mathbf{m}$ has the GSB-property. Further, we see that if $\mathbf{m}_{u,w}$ has the *- property, then $\mathbf{m}_{u,w_4}, w_4 \supset w, w_4 \in \mathcal{W}$, has the *- property, too. □

4.16. Theorem. Let $\mathbf{m}$ be a $u-\mathcal{U}, \mathcal{W}-\sigma$- additive measure. A function $\mathbf{f} : T \to \mathbf{X}$ is $\mathcal{P}_{\mathcal{U},\mathcal{W}}$- integrable if and only if there exists a couple $(u, w) \in \mathcal{U} \times \mathcal{W}$, such that
(i) $\mathbf{m}_{u,w}$ has the *- property,
(ii) there exists a sequence $\mathbf{f}_n, n \in \mathbb{N}$, of $\mathcal{P}_{u,w}$- simple functions, such that
(ii.1) the sequence $\mathbf{f}_n, n \in \mathbb{N}$, u- converges to $\mathbf{f}$ $\hat{\mathbf{m}}_{u,w}$- a.e.,
(ii.2) the sequence $\mathbf{m}_{\mathbf{f}_n}(\cdot)$, $n \in \mathbb{N}$, of (indefinite) integrals w- converges for each $E \in \sigma(\mathcal{P})$.

Proof. Cf. [2], Th. 9. □

4.17. In what follows we suppose that whenever the measure $\mathbf{m}$ is $u-\mathcal{U}, \mathcal{W}-\sigma$-additive and the function $\mathbf{f}$ is $\mathcal{P}_{u,w}$-integrable, then $\mathbf{m}_{u,w}$ has the *- property, $(u, w) \in \mathcal{U} \times \mathcal{W}$.

4.18. Lemma. Assume that $\mathbf{m}$ is a $u - \mathcal{U}, \mathcal{W} - \sigma$- additive measure and $\mathbf{m}_{u,w_1}$ has the *- property for every $w_1 \supset w, (u, w), (u, w_1) \in \mathcal{U} \times \mathcal{W}$. If the function $\mathbf{f}$ is $\mathcal{P}_{u,w}$- integrable, then $\mathbf{f}$ is also $\mathcal{P}_{u,w_1}$- integrable.

Proof. We shall verify the conditions of Theorem 4.16.
Let $w_1 \supset w,\ (u, w), (u, w_1) \in \mathcal{U} \times \mathcal{W}$.
By Lemma 3.2., if $E \in \mathcal{P}_{u,w}$, then $E \in \mathcal{P}_{u,w_1}$. This implies that if a function $\mathbf{f}$ is $\mathcal{P}_{u,w}$-simple, then $\mathbf{f}$ is $\mathcal{P}_{u,w_1}$-simple, too.
By Lemma 3.14., the $\hat{\mathbf{m}}_{u,w}$-a.e. u-convergence implies the $\hat{\mathbf{m}}_{u,w_1}$-a.e. u-convergence.
By Lemma 1.19., the w-convergence in $\mathbf{Y}$ implies the w_1-convergence in $\mathbf{Y}$.
The needed *- property is assumed. □

4.19. Definition. We say that a function $\mathbf{f}\colon T \to \mathbf{X}$ is $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$- null if there exists a couple $(u, w) \in \mathcal{U} \times \mathcal{W}$, such that $\hat{\mathbf{m}}_{u,w}(N) = 0$, where $N \in \sigma(\mathcal{P})$ and $\{t \in T;\ \mathbf{f}(t) \neq 0\} \subset N$. If the function $\mathbf{f}$ is $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$- null, then for every $E \in \sigma(\mathcal{P})$ we define

$$\int_E \mathbf{f}\ \mathrm{d}\mathbf{m} = 0.$$

4.20. Theorem. Assume that $\mathbf{m}$ is a $u - \mathcal{U}, \mathcal{W} - \sigma$- additive measure and has the GSB- property. Let $E \in \sigma(\mathcal{P})$. Let $\mathbf{h}, \mathbf{g}$ be $\mathcal{P}_{u_{\mathbf{h}},w_{\mathbf{h}}}$-, $\mathcal{P}_{u_{\mathbf{g}},w_{\mathbf{g}}}$- integrable functions (and both $\mathbf{m}_{u_{\mathbf{h}},w_{\mathbf{h}}}$, $\mathbf{m}_{u_{\mathbf{g}},w_{\mathbf{g}}}$ have the *- property),such that $\mathbf{g} + \mathbf{h} = 0$. Then

$$\int_E \mathbf{h}\ \mathrm{d}\mathbf{m} + \int_E \mathbf{g}\ \mathrm{d}\mathbf{m} = 0.$$

Proof. By Theorem 4.16., there are sequences $\mathbf{h}_n, \mathbf{g}_n, n \in \mathbb{N}$, of $\mathcal{P}_{u_{\mathbf{h}},w_{\mathbf{h}}}$-, $\mathcal{P}_{u_{\mathbf{g}},w_{\mathbf{g}}}$-, simple functions $u_{\mathbf{h}}$-, $u_{\mathbf{g}}$-converging $\mathbf{m}_{u_{\mathbf{h}},w_{\mathbf{h}}}$-, $\mathbf{m}_{u_{\mathbf{g}},w_{\mathbf{g}}}$- a.e. to $\mathbf{h}, \mathbf{g}$, respectively, and

$$\lim_{n\to\infty} p_{w_{\mathbf{h}}} \left(\int_E \mathbf{h}_n\ \mathrm{d}\mathbf{m} - \int_E \mathbf{h}\ \mathrm{d}\mathbf{m} \right) = 0,$$

$$\lim_{n\to\infty} p_{w_{\mathbf{g}}} \left(\int_E \mathbf{g}_n \, \mathrm{d}\mathbf{m} - \int_E \mathbf{g} \, \mathrm{d}\mathbf{m} \right) = 0.$$

Since

$$p_{u_{\mathbf{h}} \vee u_{\mathbf{g}}}([\mathbf{h}_n(t) + \mathbf{g}_n(t)] - [\mathbf{h}(t) + \mathbf{g}(t)]) =$$
$$= p_{u_{\mathbf{h}} \vee u_{\mathbf{g}}}([\mathbf{h}_n(t) + \mathbf{g}_n(t)])$$
$$\leq p_{u_{\mathbf{h}}}(\mathbf{h}_n(t) - \mathbf{h}(t)) + p_{u_{\mathbf{g}}}(\mathbf{g}_n(t) - \mathbf{g}(t)]),$$

the sequence $(\mathbf{h}_n + \mathbf{g}_n), n \in \mathbb{N},\ u_{\mathbf{h}} \vee u_{\mathbf{g}}$-converges everywhere on the set T except in a set $N \in \mathcal{N}(\hat{\mathbf{m}}_{u_{\mathbf{h}},w_{\mathbf{h}}}) \vee \mathcal{N}(\hat{\mathbf{m}}_{u_{\mathbf{g}},w_{\mathbf{g}}}) = \mathcal{N}(\hat{\mathbf{m}}_{u_{\mathbf{h}\wedge u_{\mathbf{g}}},w_{\mathbf{h}\vee\mathbf{g}}})$, i.e. $\hat{\mathbf{m}}_{u_{\mathbf{h}\wedge u_{\mathbf{g}}},w_{\mathbf{h}\vee\mathbf{g}}}$-a.e., to $\mathbf{h} + \mathbf{g} = 0$.

Since $\hat{\mathbf{m}}_{u_{\mathbf{h}} \vee u_{\mathbf{g}}, w_{\mathbf{h}} \vee w_{\mathbf{g}}}(T) < \infty$, cf. [2], p. 346, then $\hat{\mathbf{m}}_{u_{\mathbf{h}} \vee u_{\mathbf{g}}, w_{\mathbf{h}} \vee w_{\mathbf{g}}}(E \setminus N) < \infty$, too.

Let $\varepsilon > 0$ be given. We have:

$$p_{w_{\mathbf{h}} \vee w_{\mathbf{g}}} \left(\int_E \mathbf{h} \, \mathrm{d}\mathbf{m} + \int_E \mathbf{g} \, \mathrm{d}\mathbf{m} \right) =$$

$$= p_{w_{\mathbf{h}} \vee w_{\mathbf{g}}} \left(\int_E [\mathbf{h}_n + \mathbf{g}_n] \, \mathrm{d}\mathbf{m} - \left[\int_E \mathbf{h} \, \mathrm{d}\mathbf{m} + \int_E \mathbf{g} \, \mathrm{d}\mathbf{m} \right] - \int_E [\mathbf{h}_n + \mathbf{g}_n] \, \mathrm{d}\mathbf{m} \right)$$
$$\leq p_{w_{\mathbf{h}} \vee w_{\mathbf{g}}} \left(\int_E [\mathbf{h}_n - \mathbf{h}] \, \mathrm{d}\mathbf{m} \right) + p_{w_{\mathbf{h}} \vee w_{\mathbf{g}}} \left(\int_E [\mathbf{g}_n - \mathbf{g}] \, \mathrm{d}\mathbf{m} \right) +$$
$$+ p_{w_{\mathbf{h}} \vee w_{\mathbf{g}}} \left(\int_E [\mathbf{h}_n + \mathbf{g}_n] \, \mathrm{d}\mathbf{m} \right) \leq$$
$$\leq \varepsilon/4 + \varepsilon/4 + p_{w_{\mathbf{h}} \vee w_{\mathbf{g}}} \left(\int_{E \setminus N} [\mathbf{h}_n + \mathbf{g}_n] \, \mathrm{d}\mathbf{m} \right) + p_{w_{\mathbf{h}} \vee w_{\mathbf{g}}} \left(\int_N [\mathbf{h}_n + \mathbf{g}_n] \, \mathrm{d}\mathbf{m} \right) \leq$$
$$\leq 3\varepsilon/4 + p_{w_{\mathbf{h}} \vee w_{\mathbf{g}}} \left(\int_{E \setminus (N \cup N_{n_k})} [\mathbf{h}_n + \mathbf{g}_n] \, \mathrm{d}\mathbf{m} + \int_{N_{n_k}} [\mathbf{h}_n + g_n] \, \mathrm{d}\mathbf{m} \right) \leq$$
$$\leq 3\varepsilon/4 + \hat{\mathbf{m}}_{u_{\mathbf{h}} \vee u_{\mathbf{g}}, w_{\mathbf{h}} \vee w_{\mathbf{g}}}(E \setminus (N \cup N_{n_k})) \cdot \|\mathbf{h}_n + \mathbf{g}_n\|_{E \setminus (N \cup N_{n_k}), u_{\mathbf{h}} \vee u_{\mathbf{g}}} + \varepsilon/8 = \varepsilon,$$

where

$$p_{w_{\mathbf{h}} \vee w_{\mathbf{g}}} \left(\int_N [\mathbf{h}_n + \mathbf{g}_n] \, \mathrm{d}\mathbf{m} \right) = 0,$$

because N is a $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-null set. Further, there exists a sequence of sets $N_{n_k} \in \sigma(\mathcal{P}), N_{n_k} \subset E, n_k \in \mathbb{N}$, such that $\hat{\mathbf{m}}_{u_{\mathbf{h}} \vee u_{\mathbf{g}}, w_{\mathbf{h}} \vee w_{\mathbf{g}}}(N_{n_k}) < \varepsilon$ and the sequence $(\mathbf{f}_n + \mathbf{g}_n), n \in \mathbb{N}$, converges to $\mathbf{f}$ $u_{\mathbf{h}} \vee u_{\mathbf{g}}$-uniformly on the set $E \setminus (N \cup N_{n_k}), n_k \in \mathbb{N}$.

Therefore, we can choose $n \in \mathbb{N}$, such that

$$\|\mathbf{h}_n + \mathbf{g}_n\|_{E, u_{\mathbf{h}} \vee u_{\mathbf{g}}} \leq \frac{\varepsilon}{8 \cdot \hat{\mathbf{m}}_{u_{\mathbf{h}} \vee u_{\mathbf{g}}, w_{\mathbf{h}} \vee w_{\mathbf{g}}}(E \setminus (N \cup N_{n_k}))}.$$

The assertion $p_{u_{\mathbf{h}} \vee u_{\mathbf{g}}} \left(\int_{N_{n_k}} [\mathbf{f}_n + \mathbf{g}_n] \, \mathrm{d}\mathbf{m} \right) < \varepsilon/8$ is a consequence of the following facts. The functions $\mathbf{h}_n, \mathbf{g}_n$ are $\mathcal{P}_{u_{\mathbf{h}}, w_{\mathbf{h}}}$-, $\mathcal{P}_{u_{\mathbf{g}}, w_{\mathbf{g}}}$-simple and hence also $\mathcal{P}_{u_{\mathbf{h}}, w_{\mathbf{h}}}$--, $\mathcal{P}_{u_{\mathbf{g}}, w_{\mathbf{g}}}$-integrable for each $n \in \mathbb{N}$. By Lemma 4.18., the functions $\mathbf{h}_n, \mathbf{g}_n, n \in \mathbb{N}$, are $\mathcal{P}_{u_{\mathbf{h}}, w_{\mathbf{h}} \vee w_{\mathbf{g}}}$-, $\mathcal{P}_{u_{\mathbf{g}}, w_{\mathbf{h}} \vee w_{\mathbf{g}}}$-integrable, too. We shall show that the integral

$\int_{N_{n_k}} [\mathbf{h}_n + \mathbf{g}_n] \, \mathrm{d}\mathbf{m}$ (which exists trivially, because $N_{n_k} \in \sigma(\mathcal{P})$, and $\mathbf{h}_n + \mathbf{g}_n$ is a $\mathcal{P}_{\mathcal{U},\mathcal{W}}$-simple function for each $n \in \mathbb{N}$) is uniformly $\hat{\mathbf{m}}_{u_\mathbf{h} \vee u_\mathbf{g}, w_\mathbf{h} \vee w_\mathbf{g}}$-absolutely continuous. From Definition 4.4. we have: for every $\varepsilon > 0$ there is $\eta > 0$ such that

$$K \in \sigma(\mathcal{P}), \hat{\mathbf{m}}_{u_\mathbf{h}, w_\mathbf{h} \vee w_\mathbf{g}}(K) < \eta \Rightarrow p_{w_\mathbf{h} \vee w_\mathbf{g}} \left(\int_K \mathbf{h}_n \, \mathrm{d}\mathbf{m} \right) < \varepsilon/16.$$

Clearly $\hat{\mathbf{m}}_{u_\mathbf{h} \vee u_\mathbf{g}, w_\mathbf{h} \vee w_\mathbf{g}}(K) \geq \hat{\mathbf{m}}_{u_\mathbf{h}, w_\mathbf{h} \vee w_\mathbf{g}}(K), K \in \sigma(\mathcal{P})$, and therefore also the following implication holds:

$$K \in \sigma(\mathcal{P}), \hat{\mathbf{m}}_{u_\mathbf{h} \vee u_\mathbf{g}, w_\mathbf{h} \vee w_\mathbf{g}}(K) < \eta \Rightarrow p_{w_\mathbf{h} \vee w_\mathbf{g}} \left(\int_K \mathbf{h}_n \, \mathrm{d}\mathbf{m} \right) < \varepsilon/16.$$

Analogously,

$$K \in \sigma(\mathcal{P}), \hat{\mathbf{m}}_{u_\mathbf{h} \vee u_\mathbf{g}, w_\mathbf{h} \vee w_\mathbf{g}}(K) < \eta \Rightarrow p_{w_\mathbf{h} \vee w_\mathbf{g}} \left(\int_K \mathbf{g}_n \, \mathrm{d}\mathbf{m} \right) < \varepsilon/16,$$

and, therefore,

$$\hat{\mathbf{m}}_{u_\mathbf{h} \vee u_\mathbf{g}, w_\mathbf{h} \vee w_\mathbf{g}}(K) < \eta \Rightarrow$$

$$p_{w_\mathbf{h} \vee w_\mathbf{g}} \left(\int_K [\mathbf{h}_n + \mathbf{g}_n] \, \mathrm{d}\mathbf{m} \right) = p_{w_\mathbf{h} \vee w_\mathbf{g}} \left(\int_K \mathbf{h}_n \, \mathrm{d}\mathbf{m} + \int_K \mathbf{g}_n \, \mathrm{d}\mathbf{m} \right) \leq$$

$$\leq p_{w_\mathbf{h} \vee w_\mathbf{g}} \left(\int_K \mathbf{h}_n \, \mathrm{d}\mathbf{m} \right) + p_{w_\mathbf{h} \vee w_\mathbf{g}} \left(\int_K \mathbf{g}_n \, \mathrm{d}\mathbf{m} \right) < \varepsilon/16 + \varepsilon/16 = \varepsilon/8.$$

Putting $K = N_{n_k}$ we obtain the assertion. $\square$

Theorem 4.20. leads to the following definition of a linear integral in complete bornological locally convex spaces.

4.21. Definition. Let the measure $\mathbf{m}$ is $u - \mathcal{U}, \mathcal{W} - \sigma$- additive and has the GSB-property. We say that the function $\mathbf{f} : T \to \mathbf{X}$ is $\mathcal{P}_{\mathcal{U},\mathcal{W}}\Sigma$- integr-
able if there exist couples $(u_i, w_i) \in \mathcal{U} \times \mathcal{W}$, and functions $\mathbf{f}_i \colon T \to \mathbf{X}$, $n_i \in \mathbb{N}, i = 1, 2, \dots, I$, such that they are $\mathcal{P}_{u_i, w_i}$- integrable, $i = 1, 2, \dots, I$,
respectively, and $\mathbf{f} = \sum_{i=1}^{I} \mathbf{f}_i$.

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