Rev. Roumaine Math. Pures Appl. 38(1993), 327 –337

ON CONVERGENCES OF FUNCTIONS IN COMPLETE BORNOLOGICAL LOCALLY CONVEX SPACES

JÁN HALUŠKA

Abstract. Convergences in measure, almost everywhere, almost uniform, and relations between them are studied in the context of operator valued measures in complete bornological locally convex spaces. A Egoroff-type theorem is proved.

INTRODUCTION

We can observe that in the focus of the present new measure and integration investigations lie such theories which contain a certain compatibile collection of basic theorems, a calculus. A skeleton of this collection could be, e.g., the following quadruple: the Lebesgue dominated convergence theorem, the Fubini theorem, the Radon-Nikodým theorem, and the Egoroff theorem. This calculus enables and determines further applications of the integral in a concrete branch of mathematics such as the operator theory, harmonic analysis, field theory, differential equations, distribution theory, or dynamical systems, etc. From this viewpoint, the practical aspects prevail the theory. For instance, in mathematical physics an integral construction in locally convex spaces which is based on the net convergence of simple functions can be hardly applied. On the other hand, we can observe that stochastic integration has at the present time an real conjuncture just for the reason of its large applicability.

There are Lebesgue-type measure and integration theories in Banach spaces, cf. [4], [5]. All these integral theories, classical today, fullfil the conditions described above. There is a natural tendency to generalize integrations from Banach spaces to " higher floors". For instance, there is a question how to construct a theory of integration in locally convex spaces which are non-metrizable. Another direction is to use vector lattice structures for operator valued measure integration, cf. [3].

In [8] we have developed a new technique for complete bornological locally convex spaces and operator valued measure. Then, as an example of applicability of the technique, the Lebesgue dominated convergence theorem for a Bartle-type integral ([1]) was proved. Let us remind that the problem there was the linearity of the integral. We

---

1991 *Mathematics Subject Classification.* Primary 46 G 10, Secondary 06 F 20.

Key words and phrases. Operator valued Measure; Complete Bornological Locally Convex Spaces; Convergences in measure, almost everywhere, almost uniform; Semivariation; Sequential convergence; Lattice structure.

Supported by Grant GA-SAV 367

obtained it only for a special type of the measure and the continuity of the semivariation played a substantial role.

In the present (self-contained) article we develope further the technique from [8]. The specificity of our technique is that we work with lattices. In places where an object appears in the classical theory, e.g. a submeasure, a norm, a metric, a unit sphere, a L_p-space, a σ-ideal of null sets, etc., in our theory we work with a lattice of submeasures, norms, etc. So, we can see an interesting union of the measure and integration theory with the lattice theory in the frame of functional analysis.

Usually four types of sequential convergences of functions are associated with a measure: convergence in measure, almost everywhere convergence, almost uniform convergence, and convergence in the mean, cf. [7]. The purpose of this paper is to study the first three types of convergences in connection with an $L(\mathbf{X}, \mathbf{Y})$-valued charge(= a finitely additive measure), where $\mathbf{X}, \mathbf{Y}$ are complete bornological locally convex spaces and $L(\mathbf{X}, \mathbf{Y})$ is the space of all linear continuous operators $L : \mathbf{X} \to \mathbf{Y}$.

The convergence in mean is closely related to the kind of integral considered and, consequently, with convergence theorems for this integral. A different approach than that in [8] to integration in convex bornological spaces (not necessarily locally convex) we can find in [2].

The description of the theory of complete bornological locally convex spaces we can find in [9], [10], and [12]. For a more detail explanation the basic $L(\mathbf{X}, \mathbf{Y})$-measure set structures when both $\mathbf{X}, \mathbf{Y}$ are complete bornological locally convex spaces cf. [8]. H. Weber, cf. [13], considered these structures particularly from topological aspects. Using families of submeasures and the associated topological rings O. Lipovan, cf. [11], studied convergences of functions mentioned above from different points of view, e.g. in the setting of a set-valued integration.

1. PRELIMINARIES

Let $\mathbf{X}, \mathbf{Y}$ be two *complete bornological locally convex spaces* over the field of real or complex numbers equipped with the bornologies $\mathfrak{B}_{\mathbf{X}}, \mathfrak{B}_{\mathbf{Y}}$ with bases $\mathcal{U}, \mathcal{W}$ with marked elements $u_0 \in \mathcal{U}, w_0 \in \mathcal{W}$, respectively, cf. [8], Definitions 1.3., 1.4., and 1.11. Remind that *a Banach disk* in $\mathbf{X}$ is a set which is closed, absolutely convex and the linear span of which is a Banach space. Let the bases $\mathcal{U}, \mathcal{W}$ be chosen to consist of all $\mathfrak{B}_{\mathbf{X}}$-, $\mathfrak{B}_{\mathbf{Y}}$-bounded, separable Banach disks in $\mathbf{X}, \mathbf{Y}$, respectively. So, $\mathbf{X}$ is an inductive limit of Banach spaces $\mathbf{X}_u$,

$$\mathbf{X} = \lim \operatorname{ind}_{u \in \mathcal{U}} \mathbf{X}_u,$$

cf. [10], where $\mathbf{X}_u$ is a linear span of $u \in \mathcal{U}$ and $\mathcal{U}$ is directed by inclusion (analogously for $\mathbf{Y}$ and $\mathcal{W}$).

On $\mathcal{U}$ *the lattice operations* are defined as follows. For $u_1, u_2 \in \mathcal{U}$ we have: $u_1 \wedge u_2 = u_1 \cap u_2, u_1 \vee u_2 = \operatorname{acs}(u_1 \cup u_2)$, where acs denotes the topological closure of the absolutely convex span of the set. Analogously for $\mathcal{W}$.

Recall that the basis $\mathcal{U}$ of the bornology $\mathfrak{B}_{\mathbf{X}}$ has *a marked element* $u_0 \in \mathcal{U}, u_0 \neq \{0\}$, if the following property holds: $u, q \in \mathcal{U} \Rightarrow u \wedge q \supset u_0$.

Let $T \neq \emptyset$ be a set. Denote by 2^T the potential set of the set T and by $\Delta \subset 2^T$ a δ-ring of sets. If $\mathcal{A}$ is a system of subsets of the set T, then $\sigma(\mathcal{A})$ denotes the σ-algebra generated by the system $\mathcal{A}$. Denote by $\Sigma = \sigma(\Delta), \mathbb{N} = \{1, 2, \dots\}$.

We use χ_E to denote the characteristic function of the set E and $L(\mathbf{X},\mathbf{Y})$ the space of all continuous linear operators $L:\mathbf{X}\to\mathbf{Y}$.

By $p_u:\mathbf{X}\to[0,\infty]$ we denote Minkowski functional of the set $u\in\mathcal{U}$ (If u does not absorb $\mathbf{x}\in\mathbf{X}$, we put $p_u(\mathbf{x})=\infty$.). Similarly, p_w denotes Minkowski functional of the set $w\in\mathcal{W}$.

For every $(u,w)\in\mathcal{U}\times\mathcal{W}$, denote by $\hat{\mathbf{m}}_{u,w}$ *a* (u,w)*-semivariation* of a charge $\mathbf{m}:\Delta\to L(\mathbf{X},\mathbf{Y})$, where $\hat{\mathbf{m}}_{u,w}:\Sigma\to[0,\infty], \hat{\mathbf{m}}_{u,w}(E)=\sup p_w\left(\sum_{i=1}^{I}\mathbf{m}(E\cap E_i)\mathbf{x}_i\right)$, $E\in\Sigma$, where the supremum is taken over all finite sets $\{\mathbf{x}_i\in\mathbf{X};\ \mathbf{x}_i\in u, i=1,2,\dots,I\}$ and all disjoint sets $\{E_i\in\Delta;\ i=1,2,\dots,I\}$. It is well-known that $\hat{\mathbf{m}}_{u,w}$, is a monotone, subadditive set function, and $\hat{\mathbf{m}}_{u,w}(\emptyset)=0$ for every $(u,w)\in\mathcal{U}\times\mathcal{W}$. Denote by $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}=\{\hat{\mathbf{m}}_{u,w};\ (u,w)\in\mathcal{U}\times\mathcal{W}\}$, cf. [8], Definition 2.7 and Lemmas 2.10−2.15. A set $E\in\Sigma$ is said to be of *finite* $(\mathcal{U},\mathcal{W})$-semivariation if there exists a couple $(u,w)\in\mathcal{U}\times\mathcal{W}$, such that $\hat{\mathbf{m}}_{u,w}(E)<\infty$.

Denote by $\mathfrak{N}(\hat{\mathbf{m}}_{u,w})=\{N\in\Sigma;\ \hat{\mathbf{m}}_{u,w}(N)=0\}$, $(u,w)\in\mathcal{U}\times\mathcal{W}$, cf. [8], Lemma 3.14. A set $N\in\Sigma$ is called $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-null* if there exists a couple $(u,w)\in\mathcal{U}\times\mathcal{W}$, such that $\hat{\mathbf{m}}_{u,w}(N)=0$. We say that an assertion holds $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-almost everywhere*, shortly $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e., if it holds everywhere except in an $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-null set.

A function $\mathbf{f}:T\to\mathbf{X}$ is called Δ*-simple* if $\mathbf{f}(T)$ is a finite set and $\mathbf{f}^{-1}(\mathbf{x})\in\Delta$ for every $\mathbf{x}\in\mathbf{X}\setminus\{0\}$. The space of all Δ-simple functions is denoted by $\mathcal{S}$.

2. CONVERGENCES OF MEASURABLE FUNCTIONS

The following three definitions introduce the analogies of the notions of convergences almost everywhere, almost uniform, and in measure to the case of operator valued charges in complete bornological locally convex spaces.

Definition 2.1. Let $E\in\Sigma$.

(a) Let $r\in\mathcal{U},(u,w)\in\mathcal{U}\times\mathcal{W}$. We say that *a sequence* $\mathbf{f}_i:T\to\mathbf{X}, i\in\mathbb{N}$, *of functions* (r,E)*-converges* $\hat{\mathbf{m}}_{u,w}$-a.e. *to a function* $\mathbf{f}:T\to\mathbf{X}$ if $\lim_{i\to\infty}p_r(\mathbf{f}_i(t)-\mathbf{f}(t))=0$ for every $t\in E\setminus N$, where $N\in\mathfrak{N}(\hat{\mathbf{m}}_{u,w})$.

(b) We say that *a sequence* $\mathbf{f}_i:T\to\mathbf{X}, i\in\mathbb{N}$, *of functions* $(\mathcal{U},E)$*-converges* $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e. *to a function* $\mathbf{f}:T\to\mathbf{X}$ if there exist $r\in\mathcal{U},(u,w)\in\mathcal{U}\times\mathcal{W}$, such that the sequence $\mathbf{f}_i, i\in\mathbb{N}$, of functions (r,E)-converges $\hat{\mathbf{m}}_{u,w}$-a.e. to $\mathbf{f}$.

Definition 2.2. Let $E\in\Sigma$.

(a) Let $r\in\mathcal{U}$. We say that *a sequence* $\mathbf{f}_i:T\to\mathbf{X}, i\in\mathbb{N}$, *of functions* (r,E)-*-converges uniformly to a function* $\mathbf{f}:T\to\mathbf{X}$ if $\lim_{i\to\infty}\|\mathbf{f}_i-\mathbf{f}\|_{E,r}=0$, where $\|\mathbf{f}\|_{E,r}=\sup_{t\in E}p_r(\mathbf{f}(t))$.

(b) Let $r\in\mathcal{U},(u,w)\in\mathcal{U}\times\mathcal{W}$. We say that *a sequence* $\mathbf{f}_i:T\to\mathbf{X}, i\in\mathbb{N}$, *of functions* (r,E)*-converges* $\hat{\mathbf{m}}_{u,w}$*-almost uniformly to a function* $\mathbf{f}:T\to\mathbf{X}$ if for every $\varepsilon>0$ there exists a set $F\in\Sigma$, such that $\hat{\mathbf{m}}_{u,w}(F)<\varepsilon$ and the sequence $\mathbf{f}_i, i\in\mathbb{N}$, of functions $(r,E\setminus F)$-converges uniformly to $\mathbf{f}$.

(c) We say that *a sequence* $\mathbf{f}_i:T\to\mathbf{X}, i\in\mathbb{N}$, *of functions* $(\mathcal{U},E)$*-converges* $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-almost uniformly to a function* $\mathbf{f}:T\to\mathbf{X}$ if there exist $r\in\mathcal{U},(u,w)\in\mathcal{U}\times\mathcal{W}$, such that the sequence $\mathbf{f}_i, i\in\mathbb{N}$, of functions (r,E)-converges $\hat{\mathbf{m}}_{u,w}$-almost uniformly to $\mathbf{f}$.

Let $W \in 2^T, (u,w) \in \mathcal{U} \times \mathcal{W}$. Then we define: $\hat{\mathbf{m}}^*_{u,w}(W) = \inf_{E\in\Sigma, W\subset E} \hat{\mathbf{m}}_{u,w}(E)$.

Definition 2.3. Let $E \in \Sigma$.

(a) Let $r \in \mathcal{U}, (u,w) \in \mathcal{U} \times \mathcal{W}$. We say that *a sequence* $\mathbf{f}_i : T \to \mathbf{X}, i \in \mathbb{N}$, *of functions* $\hat{\mathbf{m}}_{u,w}(r,E)$*-converges to a function* $\mathbf{f} : T \to \mathbf{X}$ if for every $\varepsilon > 0, \delta > 0$ there exists $i_{\varepsilon,\delta} \in \mathbb{N}$, such that for every $i \geq i_{\varepsilon,\delta}, i \in \mathbb{N}$, the following inequality is true: $\hat{\mathbf{m}}^*_{u,w}(\{t \in E;\ p_r(\mathbf{f}_i(t) - \mathbf{f}(t)) \geq \delta\}) < \varepsilon$.

(b) We say that *a sequence* $\mathbf{f}_i : T \to \mathbf{X}, i \in \mathbb{N}$, *of functions* $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}(\mathcal{U},E)$*-converges to a function* $\mathbf{f} : T \to \mathbf{X}$ if there exist $r \in \mathcal{U}, (u,w) \in \mathcal{U} \times \mathcal{W}$, such that the sequence $\mathbf{f}_i, i \in \mathbb{N}$, of functions $\hat{\mathbf{m}}_{u,w}(r,E)$-converges to $\mathbf{f}$.

The following lemma explains the nature of the sequential convergences given in Definitions 2.1(b), 2.2(b), and 2.3(b).

Lemma 2.4. *Let* $E \in \Sigma$. *Let* $r, r_1 \in \mathcal{U}, (u,w), (u_1,w_1) \in \mathcal{U} \times \mathcal{W}, r \subset r_1, u \supset u_1, w \subset w_1$. *If a sequence* $\mathbf{f}_i : T \to \mathbf{X}, i \in \mathbb{N}$,

(a) (r,E)*-converges* $\hat{\mathbf{m}}_{u,w}$*-a.e.,*
(b) (r,E)*-converges* $\hat{\mathbf{m}}_{u,w}$*-almost uniformly,*
(c) $\hat{\mathbf{m}}_{u,w}(r,E)$*-converges,*

to a function $\mathbf{f} : T \to \mathbf{X}$, *then the sequence* $\mathbf{f}_i, i \in \mathbb{N}$,

(a) (r_1,E)*-converges* $\hat{\mathbf{m}}_{u_1,w_1}$*-a.e.,*
(b) (r_1,E)*-converges* $\hat{\mathbf{m}}_{u_1,w_1}$*-almost uniformly,*
(c) $\hat{\mathbf{m}}_{u_1,w_1}(r_1,E)$*-converges,*

to $\mathbf{f}$, *respectively.*

Proof. It is easy to see that $\hat{\mathbf{m}}_{u_1,w_1}(E) \leq \hat{\mathbf{m}}_{u,w}(E)$ for every $E \in \Sigma$, cf. [8], Lemma 2.12, and $p_r(\mathbf{x}) \geq p_{r_1}(\mathbf{x})$ for every $\mathbf{x} \in \mathbf{X}$. The rest of the proof follows from Definitions 2.1(a), 2.2(a), and 2.3(a), respectively.

Our basis space which we deal with is the space of all measurable functions. We introduce it with the following definition.

Definition 2.5. We use $\mathcal{M}_{\Delta,\mathcal{U}}$ to denote *the space of all measurable functions*, the largest vector space of functions $\mathbf{f} : T \to \mathbf{X}$ with the property: there exists $r \in \mathcal{U}$, such that for every $u \supset r, u \in \mathcal{U}$, and $\delta > 0$ the set $\{t \in T;\ p_u(\mathbf{f}(t)) \geq \delta\} \in \Sigma$. We say that a function $\mathbf{f}$ is $\Delta,\mathcal{U}$*-measurable* if $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$.

Note that if we suppose all dealing functions to be $\Delta,\mathcal{U}$-measurable, then $\hat{\mathbf{m}}_{u,w} = \hat{\mathbf{m}}^*_{u,w}$ for every $(u,w) \in \mathcal{U} \times \mathcal{W}$ in Definition 2.3.

Remark 2.6. It can be proved that $\mathcal{M}_{\Delta,\mathcal{U}} \supset \mathcal{F}_{\Delta,\mathcal{U}}$, where $\mathcal{F}_{\Delta,\mathcal{U}} = \{\mathbf{f} : T \to \mathbf{X}; \exists r \in \mathcal{U}, \exists \mathbf{f}_i \in \mathcal{S}, i \in \mathbb{N}, \forall t \in T : \lim_{i\to\infty} p_r(\mathbf{f}_i(t) - \mathbf{f}(t)) = 0\}$.

The remaining lemmas of this section show that the introduced convergences of function are correct.

Lemma 2.7. *Let* $E \in \Sigma$. *If a sequence* $\mathbf{f}_i : T \to \mathbf{X}, i \in \mathbb{N}$, *of functions* $(\mathcal{U},E)$*-converges* $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-a.e. to functions* $\mathbf{f} : T \to \mathbf{X}$ *and* $\mathbf{g} : T \to \mathbf{X}$, *too, then* $\mathbf{f} = \mathbf{g}$ $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$*-a.e. on* E.

Proof. By assumption, there exist $r_1 \in \mathcal{U}, N_1 \in \mathfrak{N}(\hat{\mathbf{m}}_{u_1,w_1}), (u_1, w_1) \in \mathcal{U} \times \mathcal{W}$, and $r_2 \in \mathcal{U}, N_2 \in \mathfrak{N}(\hat{\mathbf{m}}_{u_2,w_2}), (u_2, w_2) \in \mathcal{U} \times \mathcal{W}$, such that $\lim_{i\to\infty} p_{r_1}(\mathbf{f}_i(t) - \mathbf{f}(t)) = 0$ for every $t \in E \setminus N_1$ and $\lim_{i\to\infty} p_{r_2}(\mathbf{g}_i(t) - \mathbf{g}(t)) = 0$ for every $t \in E \setminus N_2$.

Then the uniqness of the bornological limit implies $\mathbf{f}(t) = \mathbf{g}(t)$ for every $t \in E \setminus (N_1 \cup N_2)$. Further, $N_1 \cup N_2 = N \in \mathfrak{N}(\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}})$. Cf. [8], Lemma 1.16, Lemma 2.12, and Lemma 2.14.

Lemma 2.8. *Let $E \in \Sigma$. If a sequence $\mathbf{f}_i \in \mathcal{M}_{\Delta,\mathcal{U}}, i \in \mathbb{N}$, of functions $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}(\mathcal{U}, E)$-- converges to functions $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$ and $\mathbf{g} \in \mathcal{M}_{\Delta,\mathcal{U}}$, too, then $\mathbf{f} = \mathbf{g}$ $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e. on $E \in \Sigma$.*

Proof. Let $\varepsilon > 0, \delta > 0$ be two arbitrary reals, $r_1, r_2 \in \mathcal{U}$, $(u_1, w_1), (u_1, w_2) \in \mathcal{U} \times \mathcal{W}$ be as in Definition 2.3 for the functions $\mathbf{f}$ and $\mathbf{g}$, respectively. Denote by $u = u_1 \wedge u_2, w = w_1 \vee w_2$, and $r = r_1 \vee r_2$. Then there exists $i_{\varepsilon,\delta} \in \mathbb{N}$, such that for every $i \geq i_{\varepsilon,\delta}$, $i \in \mathbb{N}$, we have:

$$
\begin{aligned}
&\hat{\mathbf{m}}_{u,w}(\{t \in E;\ p_r(\mathbf{f}(t) - \mathbf{g}(t)) \geq 2 \cdot \delta\}) \leq \\
&\quad \leq \hat{\mathbf{m}}_{u,w}(\{t \in E;\ p_r(\mathbf{f}(t) - \mathbf{f}_i(t)) \geq \delta\} \cup \{t \in E;\ p_r(\mathbf{f}_i(t) - \mathbf{g}(t)) \geq \delta\}) \\
&\quad \leq \hat{\mathbf{m}}_{u_1,w_1}(\{t \in E;\ p_{r_1}(\mathbf{f}(t) - \mathbf{g}(t)) \geq \delta\}) + \hat{\mathbf{m}}_{u_2,w_2}(\{t \in E;\ p_{r_2}(\mathbf{f}(t) - \mathbf{g}(t)) \geq \delta\}) \\
&\quad < 2 \cdot \varepsilon.
\end{aligned}
$$

Since $\delta > 0, \varepsilon > 0$ are arbitrary reals, there is $\hat{\mathbf{m}}_{u,w}(\{t \in E;\ p_r(\mathbf{f}(t) - \mathbf{g}(t)) \neq 0\}) = 0$, i.e. $\mathbf{f} = \mathbf{g}$ $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e. on E.

Remark 2.9. Note that Lemma 2.7 and the below Theorem 3.2 imply that if $\mathbf{f}_i \colon T \to \mathbf{X}, i \in \mathbb{N}$, is a sequence of functions which $(\mathcal{U}, E)$-converges $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-almost uniformly to $\mathbf{f} \colon T \to \mathbf{X}$ and $\mathbf{g} \colon T \to \mathbf{X}$, too, then $\mathbf{f} = \mathbf{g}$ $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e. on $E \in \Sigma$.

In the sequel of this paper we suppose all functions to be $\Delta, \mathcal{U}$-measurable.

3. RELATIONS BETWEEN CONVERGENCES OF FUNCTIONS

In the previous section we introduced some convergences on the space $\mathcal{M}_{\Delta,\mathcal{U}}$ which are generalizations of that classical notions such as almost uniform convergence, convergence almost everywhere and convergence in measure (resp. in semivariation). Concerning the theory of integration in Banach spaces we suppose these notions and relations between them to be commonly well-known, cf. [6]. The following theorems show how these relations are satisfied in the context of complete bornological locally convex spaces.

Theorem 3.1. *Let $E \in \Sigma$ be a set of finite $(\mathcal{U}, \mathcal{W})$-semivariation. If a sequence $\mathbf{f}_i \in \mathcal{M}_{\Delta,\mathcal{U}}, i \in \mathbb{N}$, of functions $(\mathcal{U}, E)$-converges $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e. to a function $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$, then the sequence $\mathbf{f}_i, i \in \mathbb{N}$, of functions $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}(\mathcal{U}, E)$-converges to $\mathbf{f}$.*

Proof. By assumption, there are $r \in \mathcal{U}, (u_1, w_1) \in \mathcal{U} \times \mathcal{W}$, such that for every $\delta > 0$ there exists $i_\delta \in \mathbb{N}$, such that for every $i \geq i_\delta, i \in \mathbb{N}$, there holds:

$$p_r(\mathbf{f}_i(t) - \mathbf{f}(t)) < \delta, t \in E \setminus N, \hat{\mathbf{m}}_{u_1,w_1}(N) = 0.$$

By assumption, there exists a couple $(u_2, w_2) \in \mathcal{U} \times \mathcal{W}$, such that $\hat{\mathbf{m}}_{u_2,w_2}(E) < \infty$. Denote by $u = u_1 \wedge u_2, w = w_1 \vee w_2$.

Then for every $i \geq i_\delta, i \in \mathbb{N}$, we have:

$$\begin{aligned}
&\hat{\mathbf{m}}_{u,w}(\{t \in E;\ p_r(\mathbf{f}_i(t) - \mathbf{f}(t)) \geq \delta\}) = \\
&\qquad =\hat{\mathbf{m}}_{u,w}(\{t \in E \setminus N;\ p_r(\mathbf{f}_i(t) - \mathbf{f}(t)) \geq \delta\}) + \hat{\mathbf{m}}_{u,w}(N) \\
&\qquad \leq \hat{\mathbf{m}}_{u_2,w_2}(\{t \in E \setminus N;\ p_r(\mathbf{f}_i(t) - \mathbf{f}(t)) \geq \delta\}) + \hat{\mathbf{m}}_{u_1,w_1}(N) \\
&\qquad =\hat{\mathbf{m}}_{u_2,w_2}(\emptyset) = 0.
\end{aligned}$$

Theorem 3.2. *Let $E \in \Sigma$. If a sequence $\mathbf{f}_i \in \mathcal{M}_{\Delta,\mathcal{U}}, i \in \mathbb{N}$, of functions $(\mathcal{U}, E)$-converges $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-almost uniformly to a function $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$, then the sequence $\mathbf{f}_i, i \in \mathbb{N}$, of functions $(\mathcal{U}, E)$-converges $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e. to $\mathbf{f}$.*

Proof. Let $\varepsilon > 0, \delta > 0$ be two arbitrary numbers. By Definition 2.2, there exist $r \in \mathcal{U}, (u, w) \in \mathcal{U} \times \mathcal{W}$, and a set $E_\varepsilon \in \Sigma$, such that $\hat{\mathbf{m}}_{u,w}(E_\varepsilon) < \varepsilon$ and the sequence $\mathbf{f}_i, i \in \mathbb{N}$, of functions $(r, E \setminus E_\varepsilon)$-converges uniformly. I.e., there exists $i_{\varepsilon,\delta} \in \mathbb{N}$, such that $p_r(\mathbf{f}_i(t) - \mathbf{f}(t)) < \delta$ for every $i \geq i_{\varepsilon,\delta},\ i \in \mathbb{N}$, and $t \in E \setminus E_\varepsilon$.

Let $\varepsilon = \frac{1}{k}, k \in \mathbb{N}$. Then $F_k = E_\varepsilon \in \Sigma, k \in \mathbb{N}$, is a sequence of sets, such that $\hat{\mathbf{m}}_{u,w}(F_k) < \frac{1}{k}$ and the sequence $\mathbf{f}_i, i \in \mathbb{N}$, of functions $(r, E \setminus F_k)$-converges uniformly for every $k \in \mathbb{N}$. Put $\bigcap_{k=1}^{\infty} F_k = F$. We have:

$$0 \leq \hat{\mathbf{m}}_{u,w}(F) \leq \hat{\mathbf{m}}_{u,w}(F_k) < \frac{1}{k} \tag{1}$$

Since $k \in \mathbb{N}$ is an arbitrary number, we conclude that $\hat{\mathbf{m}}_{u,w}(F) = 0$. Show that the sequence $\mathbf{f}_i, i \in \mathbb{N}$, of functions $(r, E \setminus F)$-converges to $\mathbf{f}$. Indeed, if $t \in E \setminus F$, then there exists $k = k(t)$, such that $t \in E \setminus F_k$. But $p_r(\mathbf{f}_i(t) - \mathbf{f}(t)) < \delta$ for every $t \in E \setminus F_k$ and $i \geq i_{\varepsilon,\delta},\ i \in \mathbb{N}$.

Theorem 3.3. *Let $E \in \Sigma$. If a sequence $\mathbf{f}_i \in \mathcal{M}_{\Delta,\mathcal{U}}, i \in \mathbb{N}$, of functions $(\mathcal{U}, E)$-converges $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-almost uniformly to a function $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$, then the sequence $\mathbf{f}_i, i \in \mathbb{N}$, of functions $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}(\mathcal{U}, E)$-converges to $\mathbf{f}$.*

Proof. Let $\varepsilon > 0, \delta > 0$, be two arbitrary positive numbers. By Definition 2.2, there exist $r \in \mathcal{U}, (u, w) \in \mathcal{U} \times \mathcal{W}$, and the set $F \in \Sigma$, such that $\hat{\mathbf{m}}_{u,w}(F) < \varepsilon$ and the sequence $\mathbf{f}_i, i \in \mathbb{N}$, of functions $(r, E \setminus F)$-converges uniformly, i.e. there is $i_0 = i_0(\varepsilon, \delta) \in \mathbb{N}$, such that $p_r(\mathbf{f}_i(t) - \mathbf{f}(t)) < \delta$ for every $t \in E \setminus F$ and $i \geq i_0, i \in \mathbb{N}$. So,

$$\{t \in E;\ p_r(\mathbf{f}_i(t) - \mathbf{f}(t)) \geq \delta\} \subset F,$$

for every $i \geq i_0$ and

$$\hat{\mathbf{m}}_{u,w}(\{t \in E;\ p_r(\mathbf{f}_i(t) - \mathbf{f}(t)) \geq \delta\}) < \varepsilon.$$

To prove the assertion of Theorem 3.5, we introduce the following notion of the σ-subadditivity of the $(\mathcal{U}, \mathcal{W})$-semivariation.

Definition 3.4. We say that a charge $\mathbf{m}$ is of *σ-subadditive* $(\mathcal{U},\mathcal{W})$-semivariation if $\hat{\mathbf{m}}_{u,w}$ is a σ-subadditive function for every $(u,w)\in\mathcal{U}\times\mathcal{W}$, i.e. if

$$E_n \in \Sigma, n\in\mathbb{N} \Rightarrow \hat{\mathbf{m}}_{u,w}\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty}\hat{\mathbf{m}}_{u,w}(E_n).$$

Theorem 3.5. *Let $E\in\Sigma$. Let a charge $\mathbf{m}$ be of σ-subadditive $(\mathcal{U},\mathcal{W})$-semivariation. If a sequence $\mathbf{f}_i \in \mathcal{M}_{\Delta,\mathcal{U}}, i\in\mathbb{N}$, of functions $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}(\mathcal{U},E)$-converges to a function $\mathbf{f}\in\mathcal{M}_{\Delta,\mathcal{U}}$, then there exists a subsequence $\mathbf{f}_{k_i}\in\mathcal{M}_{\Delta,\mathcal{U}}, i\in\mathbb{N}$, of the sequence $\mathbf{f}_i, i\in\mathbb{N}$, which $(\mathcal{U},E)$-converges $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-almost uniformly to $\mathbf{f}$.*

Proof. Let $E\in\Sigma$. By Definition 2.3, there exist $r\in\mathcal{U}, (u,w)\in\mathcal{U}\times\mathcal{W}$, such that for every $\varepsilon>0, \delta>0$, there exists $i_1 = i_1(\varepsilon,\delta)\in\mathbb{N}$, such that for every $i\ge i_1, i\in\mathbb{N}$, the following inequality is true:

$$\hat{\mathbf{m}}_{u,w}\left(\left\{t\in E;\ p_r(\mathbf{f}_i(t)-\mathbf{f}(t)) \ge \frac{\delta}{2}\right\}\right) < \frac{\varepsilon}{2}. \tag{2}$$

For every $i,j\in\mathbb{N}$, we have

$$\{t\in E;\ p_r(\mathbf{f}_i(t)-\mathbf{f}_j(t))\ge\delta\}\subset$$

$$\subset\left\{t\in E;\ p_r(\mathbf{f}_i(t)-\mathbf{f}(t))\ge\frac{\delta}{2}\right\}\cup\left\{t\in E;\ p_r(\mathbf{f}(t)-\mathbf{f}_j(t))\ge\frac{\delta}{2}\right\}. \tag{3}$$

The relations (2) and (3) imply

$$\hat{\mathbf{m}}_{u,w}(\{t\in E;\ p_r(\mathbf{f}_i(t)-\mathbf{f}_j(t))\ge\delta\})<\varepsilon \tag{4}$$

for every $i,j\ge i_1,\ \ i,j\in\mathbb{N}$.

For every $i\in\mathbb{N}$ there exists $m=m(i)\in\mathbb{N}$, such that if $k,l\ge m,\ k,l\in\mathbb{N}$, then

$$\hat{\mathbf{m}}_{u,w}\left(\left\{t\in E;\ p_r(\mathbf{f}_k(t)-\mathbf{f}_l(t))\ge\frac{1}{2^i}\right\}\right)<\frac{1}{2^i}. \tag{5}$$

So, (5) implies that there exists a subsequence $\mathbf{f}_{k_i}\in\mathcal{M}_{\Delta,\mathcal{U}}, i\in\mathbb{N}$, such that

$$\hat{\mathbf{m}}_{u,w}\left(\left\{t\in E;\ p_r(\mathbf{f}_{k_{i+1}}(t)-\mathbf{f}_{k_i}(t))\ge\frac{1}{2^i}\right\}\right)<\frac{1}{2^i}.$$

Without loss of generality suppose $\mathbf{f}_{k_i}=\mathbf{f}_i, i\in\mathbb{N}$.

Show that the sequence $\mathbf{f}_i, i\in\mathbb{N}$, of functions $(\mathcal{U},E)$-converges $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-almost uniformly. Put

$$E_i=\left\{t\in E;\ p_r(\mathbf{f}_{i+1}(t)-\mathbf{f}_i(t))\ge\frac{1}{2^i}\right\}.$$

There exists $i_2 = i_2(\varepsilon, \delta) \in \mathbb{N}$, such that $1/\left(2^{i_2-1}\right) < \varepsilon$. Put $F = \bigcup_{i=i_2}^{\infty} E_i$. By the σ-subadditivity of the $(\mathcal{U}, \mathcal{W})$-semivariation we obtain:

$$\hat{\mathbf{m}}_{u,w}(F) \leq \sum_{i=i_2}^{\infty} \hat{\mathbf{m}}_{u,w}(E_i) < \frac{1}{2^{i_2}-1} < \varepsilon.$$

We show that the sequence $\mathbf{f}_i, i \in \mathbb{N}$, of functions $(r, E \setminus F)$-converges uniformly. Choose $i_3 = i_3(\varepsilon, \delta) \in \mathbb{N}$, such that $i_3 \geq i_2$ and $1/\left(2^{i_3-1}\right) < \delta$. Then

$$p_r(\mathbf{f}_i(t) - \mathbf{f}_j(t)) \leq \sum_{n=i}^{\infty} p_r(\mathbf{f}_{n+1}(t) - \mathbf{f}_n(t)) \leq \frac{1}{2^{i-1}} \leq \frac{1}{2^{i_3-1}} < \delta$$

for every $i, j \geq i_3$ and $t \in E \setminus F$.

To prove a Egoroff-type theorem we suppose the charge $\mathbf{m}$ is of σ-subadditive and continuous $(\mathcal{U}, \mathcal{W})$-semivariation.

Definition 3.6. We say that a charge $\mathbf{m}$ is of *continuous* $(\mathcal{U}, \mathcal{W})$-semivariation if for every couple $(u, w) \in \mathcal{U} \times \mathcal{W}$,

$$E_n \supset E_{n+1}, E_n \in \Sigma, n \in \mathbb{N}, \hat{\mathbf{m}}_{u,w}(E_1) < \infty, \bigcap_{n=1}^{\infty} E_n = \emptyset \Rightarrow \hat{\mathbf{m}}_{u,w}(E_n) \searrow 0.$$

Theorem 3.7 (Egoroff). *Let $E \in \Sigma$ be a set of finite $(\mathcal{U}, \mathcal{W})$-semivariation. Let a charge $\mathbf{m}$ be of continuous and σ-subadditive $(\mathcal{U}, \mathcal{W})$-semivariation. If a sequence $\mathbf{f}_i \in \mathcal{M}_{\Delta,\mathcal{U}}, i \in \mathbb{N}$, of functions $(\mathcal{U}, E)$-converges $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-a.e. to a function $\mathbf{f} \in \mathcal{M}_{\Delta,\mathcal{U}}$, then the sequence $\mathbf{f}_i, i \in \mathbb{N}$, of functions $(\mathcal{U}, E)$-converges $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$-almost uniformly to $\mathbf{f}$.*

Proof. Let $\varepsilon > 0, \delta > 0$ be given.

By assumption, there exists a couple $(u_1, w_1) \in \mathcal{U} \times \mathcal{W}$, such that $\hat{\mathbf{m}}_{u_1,w_1}(E) < \infty$.

By assumption, there exist $r \in \mathcal{U}, (u_2, w_2) \in \mathcal{U} \times \mathcal{W}$, such that for given δ there exists $i_\delta \in \mathbb{N}$, such that for every $i \geq i_\delta, i \in \mathbb{N}$, there holds

$$p_r(\mathbf{f}_i(t) - \mathbf{f}(t)) < \delta \tag{6}$$

for every $t \in E \setminus N$, where $\hat{\mathbf{m}}_{u_2,w_2}(N) = 0$. Put $E_N = E \setminus N$.

Put $u = u_1 \wedge u_2, w = w_1 \vee w_2$. We have:

$$\begin{aligned}\hat{\mathbf{m}}_{u,w}(E) &\leq \hat{\mathbf{m}}_{u,w}(E_N) + \hat{\mathbf{m}}_{u,w}(N) \\ &\leq \hat{\mathbf{m}}_{u_1,w_1}(E_N) + \hat{\mathbf{m}}_{u_2,w_2}(N) \\ &\leq \hat{\mathbf{m}}_{u_1,w_1}(E_N) + 0 < \infty.\end{aligned}$$

For every $j, m \in \mathbb{N}$, put

$$\begin{aligned}B_{m,j} &= E_N \cap \left\{ t \in E;\ p_r(\mathbf{f}_i(t) - \mathbf{f}(t)) < \frac{1}{m} \text{ and } i \geq j \right\} \\ &= E_N \cap \bigcap_{i=j}^{\infty} \left\{ t \in E;\ p_r(\mathbf{f}_i(t) - \mathbf{f}(t)) < \frac{1}{m} \right\},\ i \in \mathbb{N}.\end{aligned} \tag{7}$$

Clearly, if $k < j$, then $B_{m,j} \subset B_{m,k}$ for every $j, k, m \in \mathbb{N}$. Define

$$E_m = \bigcup_{j=1}^{\infty} B_{m,j} \in \Sigma.$$

The sequence $E_m \setminus B_{m,j}, j \in \mathbb{N}$, tends to the void set for every $m \in \mathbb{N}$. Since the semivariation $\hat{\mathbf{m}}_{\mathcal{U},\mathcal{W}}$ is continuous, there exists an index $j_m = j_m(\varepsilon) \in \mathbb{N}$, such that for every $i \geq j_m, i \in \mathbb{N}$, there holds

$$\hat{\mathbf{m}}_{u,w}(E_m \setminus B_{m,i}) < \frac{\varepsilon}{2^m}.$$

Put

$$F = \bigcup_{m=1}^{\infty} (E_m \setminus B_{m,j_m}) \cup N. \tag{8}$$

We have by the σ-subadditivity of the (u, w)-semivariation:

$$\begin{aligned} \hat{\mathbf{m}}_{u,w}(F) =& \hat{\mathbf{m}}_{u,w}\left(\bigcup_{m=1}^{\infty} (E_m \setminus B_{m,j_m}) \cup N\right) \\ \leq& \sum_{m=1}^{\infty} \hat{\mathbf{m}}_{u,w}(E_m \setminus B_{m,j_m}) + \hat{\mathbf{m}}_{u,w}(N) \leq \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} + 0 = \varepsilon \end{aligned} \tag{9}$$

Show that the sequence $\mathbf{f}_i, i \in \mathbb{N}$, of functions $(\mathcal{U}, E \setminus F)$-converges uniformly.

Without loss of generality suppose that $\delta \leq 1$. Then (6) and (7) imply

$$\bigcup_{n=1}^{\infty} E_n = E_N. \tag{10}$$

Choose $m_0 \in \mathbb{N}$, such that $\frac{1}{m_0} < \delta$. Since $B_{m,j_m} \subset E_m$, the inequalities (8) and (10) imply:

$$\begin{aligned} E_N \setminus F =& E_N \setminus \bigcup_{m=1}^{\infty} (E_m \setminus B_{m,j_m}) = \bigcup_{n=1}^{\infty} E_n \setminus \bigcup_{m=1}^{\infty} (E_m \setminus B_{m,j_m}) \\ =& \bigcap_{m=1}^{\infty} \left(\bigcup_{n=1}^{\infty} E_n \setminus (E_m \setminus B_{m,j_m})\right) = \bigcap_{m=1}^{\infty} B_{m,j_m} \subset B_{m_0,j_{m_0}}. \end{aligned} \tag{11}$$

By the definition of the set $B_{m_0,j_{m_0}}$, for every $t \in B_{m_0,j_{m_0}}$ and $i \geq j_{m_0}$ there is

$$p_r(\mathbf{f}_i(t) - \mathbf{f}(t)) < \delta. \tag{12}$$

So, (8), (9), (11), and (12) imply that for every $\varepsilon > 0, \delta > 0$ there exists an index $j_{m_0} = j_{m_0}(\varepsilon, \delta) \in \mathbb{N}$, such that for every $i \geq j_{m_0}, i \in \mathbb{N}$,

$$p_r(\mathbf{f}_i(t) - \mathbf{f}(t)) < \delta, t \in E_N \setminus B_{m_0,i} \supset E \setminus F, \hat{\mathbf{m}}_{u,w}(F) < \varepsilon,$$

i.e. the sequence $\mathbf{f}_i, i \in \mathbb{N}$, of functions $(\mathcal{U}, E \setminus F)$-converges uniformly to $\mathbf{f}$.

References

1. R. G. Bartle, *A general bilinear vector integral*, Studia Math. **15** (1956), 337 – 352.
2. F. Bombal, *Medida e integration bornologica, (in Spanish)*, Rev. R. Acad. Ci. Madrid **75** (1981), 115 – 137.
3. R. Cristescu, *Vector integrals and operators of type (I)*, Bull. Math. Soc. Sci. Math. R. S. Roumaine **32** (1988), 205 – 209.
4. N. Dinculeanu, *Vector Measures*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1966.
5. J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Providence,, Rhode Island, 1977.
6. I. Dobrakov, *On integration in Banach spaces, I.*, Czech. Math. J. **20** (1970), 511 – 536.
7. P. R. Halmos, *Measure Theory*, Springer-Verlag, Berlin – Heidelberg – New York, 1974.
8. J. Haluška, *On lattices of set functions in complete bornological locally convex spaces*, (to appear), pp. 22, Simon Stevin.
9. H. Hogbe-Nlend, *Bornologies and Functional Analysis*, North-Holland, Amsterdam – New York – Oxford,, 1977.
10. H. Jarchow, *Locally Convex Spaces*, Teubner, Stuttgart, 1981.
11. O. Lipovan, *A theorem of Egoroff type for sequences of submeasurable set-valued function*, Bull. Math. Soc. Sci. Math. R. S. Roumanie **29** (1985), 255 – 259.
12. J. V. Radyno, *Linear Equations and Bornology.*(in Russian), Izd. Beloruskogo gosudarstvennogo universiteta,, Minsk, 1982.
13. H. Weber, *Topological Boolean rings. Decomposition of finitely additive set functions*, Pacific J. Math. **110** (1984), 471 – 495.

Mathematical Institute, Slovak Academy of Sciences,
Grešákova 6, 040 01 Košice, S l o v a k i a
E-mail address: jhaluska @ saske.sk