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ON A GOGUADZE-LUZIN'S $L(\mathbf{X},\mathbf{Y})$-MEASURE CONDITION IN LOCALLY CONVEX SPACES

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Abstract. We introduce a condition (GB) which generalized Condition (B), cf. [2]. The new condition concerns families of submeasures, semivariations of the $L(\mathbf{X},\mathbf{Y})$-valued measure, where both $\mathbf{X},\mathbf{Y}$ are two locally convex topological vector spaces, and enables us to work with nets of measurable functions. If the considered measure is atomic, then Condition (GB) is fulfilled.

Introduction

In the operator valued measure theory in Banach spaces the pointwise convergence (of sequences) implies the convergence with respect to the continuous finite semivariation $\hat{\mathbf{m}}$ of the measure $\mathbf{m} : \mathcal{S} \to L(\mathbf{X},\mathbf{Y})$, where $\mathcal{S}$ is a σ-algebra of subsets of a set $T \neq \emptyset$, and $\mathbf{X},\mathbf{Y}$ are Banach spaces, cf. [1]. If $\mathbf{X}$ fails to be metrizable, the relation between these two convergencies is quite unlike the classical situation. Namely, any of the two convergencies does not imply the other one in general, cf. Example after Definition 1.11 in [5].

For classical measure space B.F.Goguadze showed in [2], that Condition (B) is essential for these considerations. The importance of Condition (B) for sequences in the classical measure and integration theory was underlined by N. N. Luzin in his dissertation [4]. We introduce Condition (GB), see Definition 1.2, which generalized Condition (B). The new condition concerns families of submeasures and enables us to work with nets. If the measure $\mathbf{m}$ is atomic, then Condition (GB) is fulfilled.

1. Definitions

For terminology concerning the nets see [3].

Let $T \neq \emptyset$ be a set and let $\mathcal{S}$ be a σ-algebra of subsets of T . Let $\mathbf{X},\mathbf{Y}$ be two Hausdorff locally convex topological vector spaces. Let $\mathcal{P}$ and $\mathcal{Q}$ be two families of seminorms which define the topologies on $\mathbf{X}$ and $\mathbf{Y}$, respectively.

1.1. Definition. *Let $\mathbf{m} : \mathcal{S} \to L(\mathbf{X},\mathbf{Y})$ be an operator valued measure σ-additive in the strong operator topology. By the p,q-semivariation of the measure $\mathbf{m}$, cf.* [5]*, we mean the family of set functions $\hat{\mathbf{m}}_{p,q} : \mathcal{S} \to [0,\infty], q \in \mathcal{Q}, p \in \mathcal{P}$, defined as follows:*

$$\hat{\mathbf{m}}_{p,q}(E) = \sup q(\sum_{i=1}^{I} \mathbf{m}(E_i)\mathbf{x}_i), E \in \mathcal{S},$$

where the supremum is taken over all finite disjoint partitions

$$\{E_i \in \mathcal{S}; E = \bigcup_{i=1}^{I} E_i, E_i \cap E_j = \emptyset, i \neq j, i,j = 1,2,\ldots,I\},$$

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of E, and all finite sets

$$\{\mathbf{x}_i \in \mathbf{X}; p(\mathbf{x}_i) \leq 1, i = 1, 2, \dots, I\}.$$

By the p, q-variation of the measure $\mathbf{m}$ we mean the set function $\mathbf{v}_{p,q}(\mathbf{m}, .) : \mathcal{S} \to [0, \infty]$, defined by the equality

$$\mathbf{v}_{p,q}(\mathbf{m}, E) = \sup \sum_{i=1}^{I} q_p(\mathbf{m}(E_i)), E \in \mathcal{S},$$

where the supremum is taken over all finite disjoint partitions

$$\{E_i \in \mathcal{S}; E = \bigcup_{i=1}^{I} E_i, E_i \cap E_j = \emptyset, i \neq j, i, j = 1, 2, \dots, I\}$$

of E and $q_p(\mathbf{m}(E)) = \sup_{p(\mathbf{x}) \leq 1} q(\mathbf{m}(E)\mathbf{x}), p \in \mathcal{P}, q \in \mathcal{Q}$.

It is easy to prove that the p, q-(semi)variation of the measure $\mathbf{m}$ is a monotone and σ-subadditive set function, and

$$\hat{\mathbf{m}}_{p,q}(\emptyset) = 0, (\mathbf{v}_{p,q}(\mathbf{m}, \emptyset) = 0)$$

for every $p \in \mathcal{P}$ and every $q \in \mathcal{Q}$.

The set $E \in \mathcal{S}$ is said to be of the positive (semi)variation if there exist $q \in \mathcal{Q}, p \in \mathcal{P}$, such that $\hat{\mathbf{m}}_{p,q}(E) > 0(\mathbf{v}_{p,q}(\mathbf{m}, E) > 0)$. We will say that $E \in \mathcal{S}$ is a $\hat{\mathbf{m}}$-null set if $\hat{\mathbf{m}}_{p,q}(E) = 0$ for every $q \in \mathcal{Q}, p \in \mathcal{P}$.

We say that the set $E \in \mathcal{S}$ is of the finite (semi)variation of the measure $\mathbf{m}$ if to every $q \in \mathcal{Q}$ there exists a $p \in \mathcal{P}$, such that $\hat{\mathbf{m}}_{p,q}(E) < \infty, (\mathbf{v}_{p,q}(\mathbf{m}, E) < \infty)$. This relation may be different for different sets of the finite (semi)variation of the measure $\mathbf{m}$ and we will denote it shortly $q \to_E p, q \in \mathcal{Q}, p \in \mathcal{P}$.

Throughout the paper I is an arbitrary directed index set representing a direction of a net. To be more exact we will sometimes specify that the net is an I-net to indicate that I is an index set for a given net.

1.2. Definition. *We say that the the measure* $\mathbf{m}$ *satisfies Condition* (GB) *if for every $E \in \mathcal{S}$ of the finite positive variation , $q \to_E p, q \in \mathcal{Q}, p \in \mathcal{P}$, and every net of sets $E_i \in \mathcal{S}$, $E_i \subset E, i \in I$, with the property that there exist real numbers $\delta(q, p, E) > 0, q \to_E p, p \in \mathcal{P}, q \in \mathcal{Q}$, such that the following implication is true:*

$$\hat{\mathbf{m}}_{p,q}(E) \geq \delta(q, p, E), i \in I \Rightarrow \lim \sup_{i \in I} E_i \neq \emptyset.$$

We say that a set $E \in \mathcal{S}$ of the positive semivariation is an $\hat{\mathbf{m}}$-atom if every proper subset A of E is either $\emptyset$ or $A \notin \mathcal{S}$. We suppose that if the measure $\mathbf{m}$ is of finite semivariation, then there is only a countable number of different $\hat{\mathbf{m}}$-atoms in $\mathcal{S}$. We say that the measure $\mathbf{m}$ is atomic if each $E \in \mathcal{S}$ can be expressed in the form $E = \bigcup_{i=1}^{\infty} A_i$, where $A_i, i \in \mathcal{N} = \{1, 2, \dots\}$, are $\hat{\mathbf{m}}$-atoms.

2. Condition (GB) and atomic $L(\mathbf{X},\mathbf{Y})$-valued measures

2.1. Lemma. *Let $\mathbf{m} : \mathcal{S} \to L(\mathbf{X},\mathbf{Y})$ be an operator valued measure σ-additive in the strong operator topology. Let $E \in \mathcal{S}$ be a set of the positive finite variation, $q \to_E p, q \in \mathcal{Q}, p \in \mathcal{P}$, of the measure $\mathbf{m}$. Let $A_k, k \in \mathcal{N}$, be a class of $\hat{\mathbf{m}}$-atoms of the measure $\mathbf{m}$, such that $A_k \subset E, k \in \mathcal{N}$. Then*

$$\mathbf{v}_{p,q}(\mathbf{m},E) = \sum_{k=1}^{\infty} \hat{\mathbf{m}}_{p,q}(A_k),$$

where $q \to_E p, q \in \mathcal{Q}, p \in \mathcal{P}$.

Proof. Let $q \to_E p, q \in \mathcal{Q}, p \in \mathcal{P}$. Then

$$\mathbf{v}_{p,q}(\mathbf{m},E) = \sum_{k=1}^{\infty} \mathbf{v}_{p,q}(\mathbf{m},A_k) = \sum_{k=1}^{\infty} q_p(A) =$$

$$= \sum_{k=1}^{\infty} \sup_{p(\mathbf{x})\leq 1} q(\mathbf{m}(A_k)\mathbf{x}) = \sum_{k=1}^{\infty} \hat{\mathbf{m}}_{p,q}(A_k),$$

because $A_k, k \in \mathcal{N}$, are $\hat{\mathbf{m}}$-atoms. □

2.2. Theorem. *Let $\mathbf{m}$ be a (countable) atomic $L(\mathbf{X},\mathbf{Y})$-valued measure σ-additive in the strong operater topology. Then the measure $\mathbf{m}$ satisfies Condition* (GB).

Proof. Let $E \in \mathcal{S}$ be an arbitrary set of the finite variation, $q \to_E p, p \in \mathcal{P}, q \in \mathcal{Q}$. Let $E_i \in \mathcal{S}, i \in I$, be a I-net, such that there are $\delta(q,p,E) > 0, q \to_E p, p \in \mathcal{P}, q \in \mathcal{Q}$, and $\hat{\mathbf{m}}_{p,q}(E_i) \geq \delta(q,p,E)$ for every $i \in I$.

Denote $A_k, k \in \mathcal{N}$, the class of $\hat{\mathbf{m}}$-atoms of the measure $\mathbf{m}$, such that $A_k \subset E, k \in \mathcal{N}$. Clearly

$$\delta(q,p,E) \leq \sum_{k=1}^{\infty} \hat{\mathbf{m}}_{p,q}(A_k) = \mathbf{v}_{p,q}(\mathbf{m},E) < \infty.$$

To prove the assertion it is enough to show that for every cofinal J-subnet of the I-net $E_i \in \mathcal{S}, i \in I, J \subset I$, there exists an $\hat{\mathbf{m}}$-atom A, such that A is a subset of each element of a cofinal K-subnet of this J-net of sets, $K \subset J$.

Suppose, this is not true for some J-subnet. Without loss of generality let it be the I-net $E_i, i \in I$, itself. So, for every $\hat{\mathbf{m}}$-atom $A_k, k \in \mathcal{N}$, there exists an index $i_k \in I$, such that $A \not\subset E_i$ for every $i \geq i_k, i \in I$. Take real numbers $\varepsilon(q,p,E) > 0$, such that $\varepsilon(q,p,E) < \delta(q,p,E)$. Then there are non-negative integers $N(q,p,E)$, such that

$$\sum_{k=1}^{\infty} \hat{\mathbf{m}}_{p,q}(A_k) - \sum_{k=1}^{N(q,p,E)} \hat{\mathbf{m}}_{p,q}(A_k) < \varepsilon.$$

The existence of such $N(q,p,E)$ follows from the series convergence on the left hand side of the inequality.

Taking the $\hat{\mathbf{m}}$-atom A_1 we find an index $i_1 \in I$, such that $A_1 \not\subset E_i$ for every $i \geq i_1, i \in I$. Thus, from the σ-subadditivity of the p,q-semivariation for $i \geq i_1$ we get

$$\hat{\mathbf{m}}_{p,q}(E_i) \leq \sum_{k=1}^{\infty} \hat{\mathbf{m}}_{p,q}(A_k) - \hat{\mathbf{m}}_{p,q}(A_1).$$

Further, we find an index $i_2 \in I, i_2 \geq i_1$, such that $A_2 \not\subset E_i$ for every $i \geq i_2, i \in I$, and

$$\hat{\mathbf{m}}_{p,q}(E_i) \leq \sum_{k=1}^{\infty} \hat{\mathbf{m}}_{p,q}(A_k) - \hat{\mathbf{m}}_{p,q}(A_1) - \hat{\mathbf{m}}_{p,q}(A_2)$$

for every $i \geq i_2, i \in I$. Repeating this procedure by induction we can write:

$$\hat{\mathbf{m}}_{p,q}(E_i) \leq \sum_{k=1}^{\infty} (A_k) - \sum_{k=1}^{N(q,p,E)} \hat{\mathbf{m}}_{p,q}(A_k) < \varepsilon$$

for every $i \geq i_{N(q,p,E)}, i, i_{N(q,p,E)} \in I$. This contradicts with $\hat{\mathbf{m}}_{p,q}(E_i) \geq \delta(q,p,E), i \in I$. The theorem is proved. $\square$

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