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ON INTEGRATION IN LOCALLY CONVEX SPACES

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Abstract. An algebraic type of integral is introduced in locally convex spaces. Two convergence theorems are given.

1. Preliminaries

1.1. Let $T \neq \emptyset$ be a set and $\mathcal{R}$ be a ring of subsets of the set T. The set $F \subset T$ is called locally measurable if there holds the implication $E \in \mathcal{R} \Rightarrow F \cap E \in \mathcal{R}$. Denote $\mathcal{R}_{loc}$ the set of all locally measurable sets. $\mathcal{R}_{loc}$ is an algebra of sets.

1.2. Let $\mathbf{X}, \mathbf{Y}$ be two vector spaces over the field $\mathcal{K}$, where $\mathcal{K}$ is the field of all real or complex numbers. Let $\mathcal{P}$, $\mathcal{Q}$ be two directed families of seminorms which define locally convex Hausdorff topologies on $\mathbf{X}$ and $\mathbf{Y}$ respectively. We suppose $\mathbf{Y}$ to be complete. Let $\mathcal{B}$ be a vector bornology on $\mathbf{X}$ and Θ be a basis of $\mathcal{B}$ which consists of circled sets in $\mathbf{X}$, cf. [3].

1.3. Denote $\mathcal{Z}$ a locally convex subspace of the space $L(\mathbf{X}, \mathbf{Y})$ of all continuous linear operators $\mathbf{z} : \mathbf{X} \to \mathbf{Y}$ equipped with the topology of the uniform convergence on elements $\theta \in \Theta$, cf. [4].

A set function $\mathbf{m} : \mathcal{R} \to \mathcal{Z}$ is called a charge (= a finitely additive vector measure) if $\mathbf{m}(\emptyset) = O$ and the following implication holds:

$$E, F \in \mathcal{R}, E \cap F = \emptyset \Rightarrow \mathbf{m}(E \cup F) = \mathbf{m}(E) + \mathbf{m}(F).$$

1.4. A function $\mathbf{f} : T \to \mathbf{X}$ is called to be simple if $\mathbf{f}(T)$ is a finite subset of $\mathbf{X}$ and for each $\mathbf{x} \in \mathbf{X} \setminus \{O\}$ there is $\mathbf{f}^{-1}(\mathbf{x}) \in \mathcal{R}$. Denote Φ_{sim} the space of all simple functions. By an indefinite integral of the function $\mathbf{f} \in \Phi_{sim}$ with respect to the charge $\mathbf{m}$ we mean

$$\int_E \mathbf{f} \, \mathrm{d}\mathbf{m} = \sum_{\mathbf{x} \in \mathbf{f}(T) \setminus \{O\}} \mathbf{m}(E \cap \mathbf{f}^{-1}(\mathbf{x}))\mathbf{x} = \int_T \mathbf{f} \chi_E \, \mathrm{d}\mathbf{m}, \tag{1}$$

where χ_E denotes the characteristic function of the set $E \in \mathcal{R}_{loc}$.

1.5. By a q,θ-semivariation $\hat{\mathbf{m}}_{q,\theta}$ of the charge $\mathbf{m}$ we mean

$$\hat{\mathbf{m}}_{q,\theta}(E) = \sup q \left(\int_E \mathbf{f} \, \mathrm{d}\mathbf{m} \right), \tag{2}$$

where $E \in \mathcal{R}_{loc}$ and the supremum is taken over the all functions $\mathbf{f} \in \Phi_{sim}$, such that $\|\mathbf{f}\|_{E,\theta} \le 1$, where

$$\|\mathbf{f}\|_{E,\theta} = \sup_{t \in E} p_\theta(\mathbf{f}(t)), \tag{3}$$

where p_θ denotes the Minkowski functional of the set $\theta \in \Theta$ (if $\theta \in \Theta$ does not absorb the point $\mathbf{x} \in \mathbf{X}$, then we put $p_\theta(\mathbf{x}) = \infty$).1

We say that the charge $\mathbf{m}$ is of finite Θ-semivariation on $E \in \mathcal{R}_{loc}$ if for every $q \in \mathcal{Q}$ there exists a set $\theta \in \Theta$, such that $\mathbf{m}(E) < \infty$.

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1.6. Lemma. *The set function $\hat{\mathbf{m}}_{q,\theta}, q \in \mathcal{Q}, \theta \in \Theta$, is a monotone, subadditive set function on $\mathcal{R}_{loc}$ and $\hat{\mathbf{m}}_{q,\theta}(\emptyset) = 0$ for every $q \in \mathcal{Q}, \theta \in \Theta$. If $E \in \mathcal{R}_{loc}, \mathbf{f} \in \Phi_{sim}, q \in \mathcal{Q}, \theta \in \Theta$, then*

$$q\left(\int_E \mathbf{f} \,\mathrm{d}\mathbf{m}\right) \leq \|\mathbf{f}\|_{E,\theta} \,.\, \hat{\mathbf{m}}_{q,\theta}(E). \tag{4}$$

1.7. It is technically convenient to extend the definition of $\hat{\mathbf{m}}_{q,\theta}, q \in \mathcal{Q}, \theta \in \Theta$, to arbitrary subset of T . We do this as follows: if W is an arbitrary subset of T the we define

$$\hat{\mathbf{m}}^*_{q,\theta}(W) = \inf_{E \in \mathcal{R}_{loc}, W \subset E} \hat{\mathbf{m}}_{q,\theta}(E). \tag{5}$$

1.8. Definition. *We say that the net of charges $(\nu_i)_{i \in I}, \nu_i : \mathcal{R}_{loc} \to \mathbf{Y}, i \in I$, is eventually equicontinuous with respect to the Θ-semivariation, if for every $q \in \mathcal{Q}, \varepsilon > 0$, there are $\theta = \theta(q) \in \Theta, i_1 = i_1(q, \theta, \varepsilon) \in I$, and $E = E(q, \theta, \varepsilon) \in \mathcal{R}_{loc}$, such that $\hat{\mathbf{m}}_{q,\theta}(E) < \infty$ and for every $i \geq i_1, i \in I$, and $D \subset T \setminus E, D \in \mathcal{R}_{loc}$, there holds: $q(\nu_i(D)) < \varepsilon$.*

1.9. Definition. *We say that the net of charges $(\nu_i)_{i \in I}, \nu_i : \mathcal{R}_{loc} \to \mathbf{Y}, i \in I$, is eventually uniformly absolutely continuous with respect to the Θ-semivariation, if for every $q \in \mathcal{Q}, \varepsilon > 0$, there are $\theta = \theta(q) \in \Theta, i_2 = i_2(q, \theta, \varepsilon) \in I$, and $\eta = \eta(q, \theta, \varepsilon) > 0$, such that for every $i \geq i_2, i \in I$, the following implication holds:*

$$A \in \mathcal{R}_{loc}, \hat{\mathbf{m}}_{q,\theta}(A) < \eta \Rightarrow q(\nu_i(A)) < \varepsilon. \tag{6}$$

1.10. Definition. *We say that the net of functions $(\mathbf{f}_i)_{i \in I}, \mathbf{f}_i : T \to \mathbf{X}, i \in I$, converges with respect to the Θ -semivariation to the function $\mathbf{f} : T \to \mathbf{X}$, if for every $q \in \mathcal{Q}, \delta > 0, \eta > 0$, there are $\theta = \theta(q) \in \Theta$ and $i_3 = i_3(q, \theta, \delta, \eta) \in I$, such that for every $i \geq i_3, i \in I$, the following implication holds:*

$$\hat{\mathbf{m}}^*_{q,\theta}(\{t \in T; p_\theta(\mathbf{f}_i(t) - \mathbf{f}(t)) \geq \delta\}) < \eta. \tag{7}$$

1.11. Definition. *We say that the function $\mathbf{f} \in \Phi_{sim}$ is Θ-integrable, we write $\mathbf{f} \in \Phi^{\Theta}_{sim}$, if for every $\mathbf{x} \in f(T) \setminus \{O\}$ the set $\mathbf{f}^{-1}(\mathbf{x})$ is of the finite Θ-semivariation.*

We say that the function $\mathbf{f} : T \to \mathbf{X}$ is Θ-measurable if it belongs to the closure of the space Φ^{Θ}_{sim} with respect to the topology of the convergence with respect to the Θ-semivariation.

We say that the Θ-measurable function $\mathbf{f} : T \to \mathbf{X}$ is Θ-integrable over T, we write $\mathbf{f} \in \Phi^{\Theta}_{int}$, if there exists a net $(\mathbf{f}_i)_{i \in I}$ in Φ^{Θ}_{sim} satisfying the following conditions:

(a) the net $(\mathbf{f}_i)_{i \in I}$ converges with respect to the Θ-semivariation to the function $\mathbf{f}$,

(b) the net $(\mathbf{m}_{\mathbf{f}_i})_{i \in I}$ is eventually uniformly absolutely continuous with respect to the Θ-semivariation,

(c) the net $(\mathbf{m}_{\mathbf{f}_i})_{i \in I}$ is eventually equicontinuous with respect to the Θ-semivariation,

(d) for every $q \in \mathcal{Q}$ there is $\theta = \theta(q) \in \Theta$ the same simultaneously in (a), (b), (c).

If $E \in \mathcal{R}_{loc}$ *then the limit*

$$\int_E \mathbf{f} \, \mathrm{d}\mathbf{m} = \lim_{i \in I} \int_E \mathbf{f}_i \, \mathrm{d}\mathbf{m} \tag{8}$$

is called an indefinite integral $\mathbf{m_f}$ *at the set* E*.*

1.12. It can be proved that the value of the integral in (8) is independent of the net of simple functions $(\mathbf{f}_i)_{i \in I}$ in this definition.

1.13. Definition. *We say that the net of simple (*Θ*-integrable) functions* $(\mathbf{f}_i)_{i \in I}$ *is fundamental (converges) in mean if the net of charges* $(\mathbf{m}_{\mathbf{f}_i})_{i \in I}$ *is fundamental (converges) in* $\mathbf{Y}$ *uniformly for every* $E \in \mathcal{R}_{loc}$*, where* $\mathbf{m_f}(.) = \int_. \mathbf{f} \, \mathrm{d}\mathbf{m} : \mathcal{R}_{loc} \to \mathbf{Y}, \mathbf{f} \in \Phi_{sim}$*, (*$\Phi_{int}^{\Theta}$*).*

2. Convergence theorems

2.1. Theorem (Vitali**).** *Let* $(\mathbf{f}_i)_{i \in I}$ *be a net in* Φ_{int}^{Θ} *and* $\mathbf{f} : T \to \mathbf{X}$ *be a function, such that the condition (a), (b), (c), and (d) in Definition 1.11. are satisfied. Then* $\mathbf{f} \in \Phi_{int}^{\Theta}$ *and the net* $(\mathbf{f}_i)_{i \in I}$ *converges in mean to the function* $\mathbf{f}$*.*

Proof. Consider first the case when $\mathbf{f}_i \in \Phi_{sim}^{\Theta}, i \in I$. In this case clearly $\mathbf{f} \in \Phi_{int}^{\Theta}$ by Definition 1.11. Since $\mathbf{Y}$ is complete, we have only to show that the net $(\mathbf{f}_i)_{i \in I}$ is fundamental in mean, i.e. we prove that the integral (8) is defined well.

Let $F \in \mathcal{R}_{loc}, E \in \mathcal{R}, q \in \mathcal{Q}$. We have:

$$d = q\left(\int_F \mathbf{f}_i \, \mathrm{d}\mathbf{m} - \int_F \mathbf{f}_j \, \mathrm{d}\mathbf{m}\right) =$$

$$q\left(\int_{F \cap (T \setminus E)} \mathbf{f}_i \, \mathrm{d}\mathbf{m} - \int_{F \cap (T \setminus E)} \mathbf{f}_j \, \mathrm{d}\mathbf{m} + \int_{F \cap E} (\mathbf{f}_i - \mathbf{f}_j) \, \mathrm{d}\mathbf{m}\right), \tag{9}$$

where $i, j \in I$. Clearly $F \cap (T \setminus E) \subset T \setminus E$ and $F \cap (T \setminus E) \in \mathcal{R}_{loc}$.

Let ε be an arbitrary positive number. Choose $E \in \mathcal{R}, \theta \in \Theta, i_1 \in I$, such as in Definition 1.8. Put $D = F \cap (T \setminus E)$. Then by Definition 1.8. we obtain for every $i, j \geq i_1$:

$$d < 2\varepsilon + q\left(\int_{F \cap E} (\mathbf{f}_i - \mathbf{f}_j) \, \mathrm{d}\mathbf{m}\right). \tag{10}$$

By (d) to given $q \in \mathcal{Q}$ let us consider the same $\theta \in \Theta$ as in Definition 1.8. (4) implies:

$$q\left(\int_{F \cap E} \mathbf{f} \, \mathrm{d}\mathbf{m}\right) \leq \|\mathbf{f}\|_{E \cap F, \theta} \, . \, \hat{\mathbf{m}}_{q,\theta}(E \cap F), \tag{11}$$

where $\mathbf{f} \in \Phi^{\Theta}_{sim}$. If $\mathbf{f}_i, \mathbf{f}_j \in \Phi^{\Theta}_{sim}, i, j \in I$, then $\mathbf{f}_i - \mathbf{f}_j \in \Phi^{\Theta}_{sim}$, too, and (11) implies for every $i, j \in I$:

$$q\left(\int_{F\cap E}(\mathbf{f}_i - \mathbf{f}_j)\ \mathrm{d}\mathbf{m}\right) \le \|\mathbf{f}_i - \mathbf{f}_j\|_{E\cap F,\theta}\ .\ \hat{\mathbf{m}}_{q,\theta}(E\cap F). \tag{12}$$

Since the charge $\mathbf{m}$ is of finite Θ-semivariation on $E \in \mathcal{R}$ and $\hat{\mathbf{m}}_{q,\theta}$ is a monotone set function, there is $\hat{\mathbf{m}}_{q,\theta}(E\cap F) < \infty$, too. Then for given $\varepsilon > 0$ there is $\delta > 0$, such that the following implication is true:

$$\mathbf{f}_i, \mathbf{f}_j \in \Phi^{\Theta}_{sim}, \|\mathbf{f}_i - \mathbf{f}_j\|_{E\cap F,\theta} < \delta \Rightarrow q\left(\int_{F\cap E}(\mathbf{f}_i - \mathbf{f}_j)\ \mathrm{d}\mathbf{m}\right) < \varepsilon. \tag{13}$$

Denote $G = \{t \in F\cap E; p_\theta(\mathbf{f}_i(t) - \mathbf{f}_j(t)) < \delta\}$. Since $\mathbf{f}_i, \mathbf{f}_j \in \Phi^{\Theta}_{sim}, i, j \in I$, there is $G \in \mathcal{R}_{loc}$. We have:

$$q\left(\int_{F\cap E}(\mathbf{f}_i - \mathbf{f}_j)\ \mathrm{d}\mathbf{m}\right) \le$$

$$q\left(\int_{(F\cap E)\cap G}(\mathbf{f}_i - \mathbf{f}_j)\ \mathrm{d}\mathbf{m}\right) + q\left(\int_{(F\cap E)\setminus G}(\mathbf{f}_i - \mathbf{f}_j)\ \mathrm{d}\mathbf{m}\right). \tag{14}$$

So, we obtain from (10), (13), and (14):

$$d < 3\varepsilon + q\left(\int_{(F\cap E)\setminus G}(\mathbf{f}_i - \mathbf{f}_j)\ \mathrm{d}\mathbf{m}\right). \tag{15}$$

By (b) the net $(\mathbf{m}_{\mathbf{f}_i})_{\mathbf{f}_i\in I}$ is eventually uniformly absolutely continuous with respect to the Θ-semivariation, see Definition 1.9. By (d) to given $q \in \mathcal{Q}$ let us consider the same $\theta \in \Theta$ as in Definition 1.8. Choose $i_2 \ge i_1$. Further, if $q(\mathbf{m}_{\mathbf{f}_i})(A) < \varepsilon, A \in \mathcal{R}_{loc}, i \in I, i \ge i_2$, then

$$q(\mathbf{m}_{\mathbf{f}_i - \mathbf{f}_j})(A) < 2\varepsilon \tag{16}$$

for $i \ge i_2, i \in I$.

By (a) the net $(\mathbf{f}_i)_{i\in I}$ converges with respect to the Θ-semivariation to $\mathbf{f}$, see Definition 1.10. Since $\hat{\mathbf{m}}_{q,\theta}, q \in \mathcal{Q}, \theta \in \Theta$, is a monotone set function, then also the net $(\mathbf{f}_i\chi_A)_{i\in I}$ converges with respect to the Θ-semivariation to $\mathbf{f}\chi_A, A \in \mathcal{R}_{loc}$, i.e. for every $i \ge i_3, i, i_3 \in I$, there is

$$\hat{\mathbf{m}}_{q,\theta}(\{t \in A; p_\theta(\mathbf{f}_i(t) - \mathbf{f}_i(t)) \ge \delta\}) < \eta. \tag{17}$$

Choose $i_3 \ge i_2$ and put $A = (F\cap E)\setminus G \in \mathcal{R}_{loc}$. Since $\mathbf{f}_i, \mathbf{f}_j \in \Phi^{\Theta}_{sim}, i, j \in I$, there is $\{t \in A; p_\theta(\mathbf{f}_i(t) - \mathbf{f}_j(t)) \ge \delta\} \in \mathcal{R}_{loc}$. In this case clearly $\hat{\mathbf{m}}^*_{q,\theta} = \hat{\mathbf{m}}_{q,\theta}$. Then (6), (15), (16), and (17) imply that for every $F \in \mathcal{R}_{loc}, q \in \mathcal{Q}, \varepsilon > 0$, there is $i_3 \in I$, such that for every $i \ge i_3, i \in I$, there is $d < 5\varepsilon$.

Let us consider $\mathbf{f}_i \in \Phi^{\Theta}_{int}, i \in I$. Let $\mathcal{I}$ denotes the family of all directed sets. By definition to every Θ-integrable function there exists a net of functions $(\mathbf{f}_{i,j})_{j \in I_i}$ in Φ^{Θ}_{sim}, such that the conditions of Definition 1.11. (a), (b), (c), and (d) are satisfied.

Consider the product of the family of directed sets $I \times \mathcal{I}^I$. An partial ordering on this product we define as the lexicographical ordering, as follows: $(i_1, j_1) \leq (i_2, j_2) \Leftrightarrow$ [either $(i_1 < i_2)$ or $(i_1 = i_2 \wedge j_1 \leq j_2)$], for $j_1 \in J_{i_1}, j_2 \in J_{i_2}, i_1, i_2 \in I$. It is easy to see that a subnet of the net $(\mathbf{f}_{i,j})_{j \in I_j}, (i,j) \in I \times \mathcal{I}^I$, can be choosen to satisfy the conditions (a), (b), (c), and (d) of this theorem. □

2.2. Theorem (Lebesgue). *Let $(\mathbf{f}_i)_{i \in I}$ be a net in $\Phi^{\Theta}_{int}, \mathbf{f} : T \to \mathbf{X}$ be a function, $E \in \mathcal{R}_{loc}$, and*

(a) the net $(\mathbf{f}_i)_{i \in I}$ converges with respect to the Θ-semivariation to $\mathbf{f}$,

(b) there is a function $\mathbf{g} \in \Phi^{\Theta}_{int}$ such that the net $(\mathbf{m}_{\mathbf{f}_i})_{i \in I}$ is such that

$$q\left(\int_E \mathbf{f}_i \, \mathrm{d}\mathbf{m}\right) \leq q\left(\int_E \mathbf{g} \, \mathrm{d}\mathbf{m}\right) \tag{18}$$

for every $i \in I$ and $q \in \mathcal{Q}$. Then $\mathbf{f} \in \Phi^{\Theta}_{int}$ and the net $(\mathbf{f}_i)_{i \in I}$ converges in mean to the function $\mathbf{f}$.

Proof. From the Theorem 2.1 there immediately follows this version of Lebesgue Dominated Convergence Theorem as a consequence, cf. also [1], [5], and [6]. □

References

[1] BARTLE R.G., *A general bilinear vector integral*, Studia Math. **15** (1956), 337 - 352.

[2] MARIK W., *Monoidwertige Integrale*, Czech. Math. J **37** (1987), 351 -375.

[3] RADYNO J.V., *Linear equations and the bornology. (in Russian)*, Izd. Bel. gosud. univ., Minsk, 1982.

[4] ROBERTSON A.P. - ROBERTSON W.J., *Topological Vector Spaces*, Cambridge University Press, Cambridge, 1964.

[5] MASSÉ C., *Intégration dans les semi-groupes*, Studia Math. **66** (1979), 57 -80.

[6] SMITH W., *Convergence in measure of integrands*, Rev. Roum. Pures et Appl. **26** (1981), 899 - 903.

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