Constraints in Universal Algebra

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Lecture 3

Outline

Lecture 1: Intersection problems and congruence SD($\wedge$) varieties

Lecture 2: Constraint problems in ternary groups (and generalizations)

Lecture 3: Constraint problems in Taylor varieties

WARNING

This lecture has been modified
to fit our shorter attention spans.

Review

An **instance of 3-CSP(A)** of degree n is a list $(s_1, C_1), \ldots, (s_p, C_p)$ where

- Each scope s_i satisfies $s_i \subseteq \{1, 2, \ldots, n\}$ and $1 \leq |s_i| \leq 3$.
- Each constraint relation C_i is a non-empty subuniverse of $\mathbf{A}^{s_i}$.

It is **3-minimal** if

- Every 3-element subset of $\{1, 2, \ldots, n\}$ occurs as a scope.
- For any two constraints $(s, C_i), (t, C_j)$, if $s \subseteq t$ then $\mathrm{proj}_s(C_j) = C_i$.

The **solution-set** of the instance is $[\![s_1, C_1]\!] \cap \cdots \cap [\![s_p, C_p]\!]$, where

$$[\![s_i, C_i]\!] \ = \ \{\mathbf{a} \in A^n \ : \ \mathrm{proj}_{s_i}(\mathbf{a}) \in C_i\} \ \leq \ \mathbf{A}^n.$$

3-CSP(A): Given a (3-minimal) instance, does a solution exist?

Central problem of CSP (Feder, Vardi) – Dichotomy

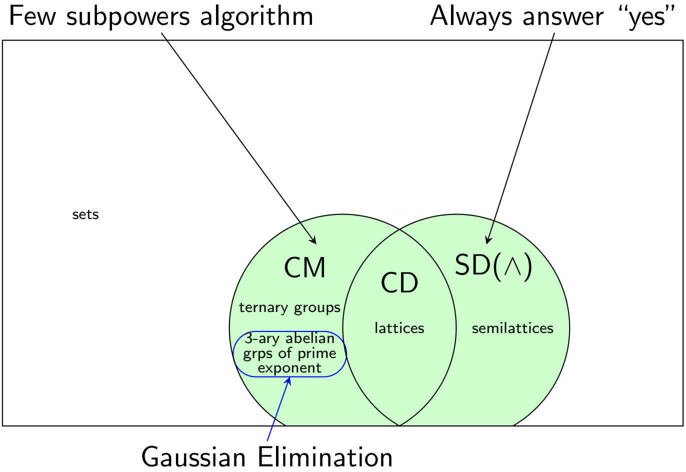
Given a finite algebra $\mathbf{A}$, either

1. Find a poly-time algorithm deciding 3-minimal instances of 3-CSP($\mathbf{A}$),

or

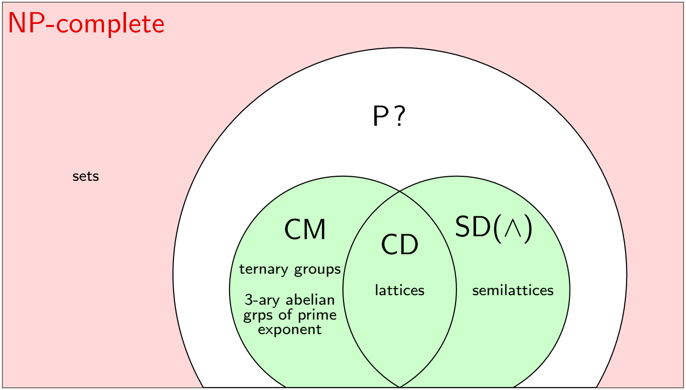
2. Show that 3-CSP($\mathbf{A}$) is NP-complete.

3 algorithms deciding 3-minimal instances of 3-CSP($\mathbf{A}$)



Algebraic CSP Dichotomy Conjecture

There is a class, outside of which each 3-CSP(**A**) is provably NP-complete.



Conjecture (Bulatov *et al*): For every **A** inside the class, 3-CSP(**A**) is in P.

Defining the "dividing line"

Definition. A term $t(x_1, \ldots, x_n)$ is a **Taylor term** (for an algebra) if

1. It is idempotent (i.e., $t(x, x, \ldots, x) = x$).
2. $n \geq 2$.
3. For each $1 \leq i \leq n$ there is an identity satisfied by t of the form

$$t(\ldots, \underset{i}{x}, \ldots) = t(\ldots, \underset{i}{y}, \ldots)$$

   where x occurs at position i on the left, y occurs at position i on the right, and all other positions are filled with x or y.

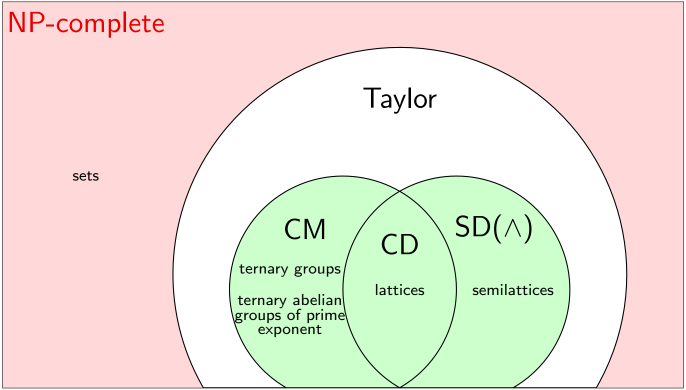
Example: A Maltsev term is a Taylor term, because $m(x, x, x) = x$ and

$$\begin{aligned} m(\underline{x}, \underline{x}, y) &= m(\underline{y}, \underline{y}, y) && \text{works for } i = 1, 2 \\ m(x, x, \underline{x}) &= m(x, y, \underline{y}) && \text{works for } i = 3 \end{aligned}$$

Theorem/Conjecture (Bulatov, Jeavons, Krokhin 2005)

Let **A** be a finite, idempotent algebra.

1. (Theorem) If **A** does *not* have a Taylor term, then 3-CSP(**A**) is NP-complete.
2. (Conjecture) Otherwise 3-CSP(**A**) is in P.



Goals of this lecture:

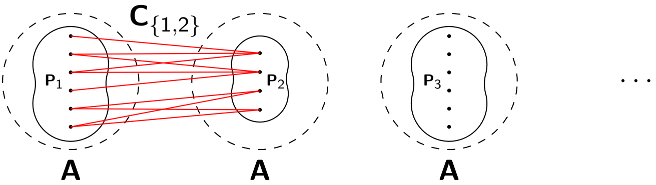
1. Describe a new, "easy" poly-time CSP algorithm for ternary groups.
   - Roughly speaking, "enforcing 3-minimality + Gaussian elimination."
2. Describe how the algorithm adapts to any Taylor algebra!
3. Caveats
   - The algorithm is for 2-CSP($\mathbf{A}$) only. (Which is fine.)
   - I don't know whether the algorithm actually works ...

First, some technicalities

Potatoes

Let $Inst = ((s_1, C_1), \ldots, (s_p, C_p))$ be a 3-minimal instance of 3-CSP($\mathbf{A}$), of degree n.

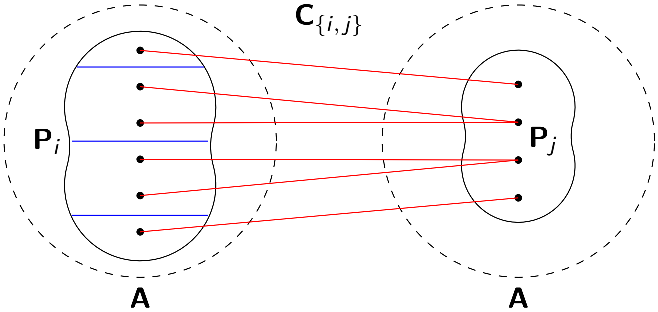
1. $V := \{1, 2, \ldots, n\}$. ("variables")
2. Every 3-element subset $s \subseteq V$ is the scope of a *unique* constraint. Call it (s, C_s).
3. For all $t \subseteq V$ with $|t| = 1$ or 2, there is a unique "implied" constraint (t, C_t), namely $(t, \mathrm{proj}_t(C_s))$ for any $t \subseteq s$ with $|s| = 3$.
4. Each $C_{\{i\}}$ is a subuniverse of $\mathbf{A}$. The corresponding subalgebra is denoted $\mathbf{P}_i$ and is called a "potato."



Congruence completeness

Definition. A 3-minimal instance of 3-CSP($\mathbf{A}$) is **congruence complete** if for every $i \in V$ and every $\alpha \in \mathrm{Con}(\mathbf{P}_i)$ there exists $j \in V$ such that $C_{\{i,j\}}$ is the graph of a surjective homomorphism $h_{ij} : \mathbf{P}_i \to \mathbf{P}_j$ with kernel α.

(I will say $\mathbf{P}_j$ "$=$" $\mathbf{P}_i/\alpha$.)



We can always enforce congruence completeness (by adding new variables).

Now we focus on ternary groups

Definition. Let $\mathbf{A} = (G, xy^{-1}z)$ be a ternary group, $\alpha \in \mathrm{Con}(\mathbf{A})$, and p a prime. We say α is an **elementary p-abelian** congruence if $N := 1/\alpha$ $(\lhd G)$ is an abelian group of exponent p.

Key fact. If $\alpha \in \mathrm{Con}(\mathbf{A})$ is elementary p-abelian, then every α-block C, considered as a subalgebra $\mathbf{C} \leq \mathbf{A}$, is a ternary abelian group of exponent p.

Proposition

Let $\mathbf{A} = (G, xy^{-1}z)$ be a finite ternary group and α a minimal congruence. If α is abelian, then α is elementary p-abelian for some prime p.

Proof. $N = 1/\alpha$ is a minimal normal subgroup of G and is abelian.

If $\exp(N) = mk$ is composite, then $\{x \in N : mx = 1\}$ is a proper nontrivial subgroup of N.

It is also characteristic in N, so is normal in G, contradiction. □

Warning: technicalities ahead

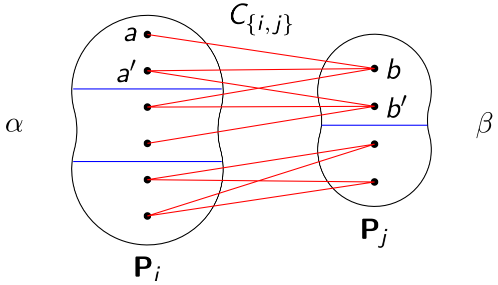
Let *Inst* be a 3-minimal instance of 3-CSP($\mathbf{A}$), of degree n.

Definition. For each prime p, let

$$VC_p = \{(i, \alpha) : i \in V, \alpha \in \mathrm{Con}(\mathbf{P}_i), \text{ and } \alpha \text{ is elementary } p\text{-abelian}\}.$$

For $(i, \alpha), (j, \beta) \in VC_p$, define $\boxed{(i, \alpha) \leq (j, \beta)}$ iff

$$\big(\ (a, b), (a', b') \in C_{\{i,j\}} \ \&\ (a, a') \in \alpha\ \big) \quad \Longrightarrow \quad (b, b') \in \beta.$$



($C_{\{i,j\}}$ "induces" a homomorphism $\mathbf{P}_i/\alpha \to \mathbf{P}_j/\beta$.)

Fix p and $(i,\alpha) \in VC_p$. Define

$$V_{(i,\alpha)} \;=\; \{ j \in V \;:\; \exists\, \beta \in \mathrm{Con}(\mathbf{P}_j) \text{ with } (i,\alpha) \le (j,\beta) \in VC_p \}.$$

Fact: for each $j \in V_{(i,\alpha)}$ there exists a smallest witnessing β; call it β_j.

Let $f_j : \mathbf{P}_i/\alpha \to \mathbf{P}_j/\beta_j$ be the homomorphism induced by $C_{\{i,j\}}$.

Definition

Let $\mathbf{A}$, *Inst*, p and (i,α) be as above.

1. $Inst_{(i,\alpha)}$ is the restriction of *Inst*[a] to the variable-set $V_{(i,\alpha)}$.
2. For each α-block B, $Inst^B_{(i,\alpha)}$ is the restriction of $Inst_{(i,\alpha)}$ obtained by
   - Replacing each potato P_j by $f_j(B)$, and
   - Restricting the constraint relations of $Inst_{(i,\alpha)}$ to these new potatoes.

[a] More precisely, of the implied constraints (t, C_t), $1 \le |t| \le 3$, of *Inst*.

Note: Each potato of $Inst^B_{(i,\alpha)}$ is a ternary abelian group of exponent p.

Lemma

Suppose $s \subseteq V_{(i,\alpha)}$ with $|s| \le 3$, and $\mathbf{c} \in C_s$. If there exists $\mathbf{a} \in \mathrm{Sol}(\mathit{Inst})$ with $\mathrm{proj}_s(\mathbf{a}) = \mathbf{c}$, then for some α-block B there exists $\mathbf{b} \in \mathrm{Sol}(\mathit{Inst}^B_{(i,\alpha)})$ with $\mathrm{proj}_s(\mathbf{b}) = \mathbf{c}$.

Proof. Given $\mathbf{a} \in \mathrm{Sol}(\mathit{Inst})$, let $B = a_i/\alpha$ and put $\mathbf{b} = \mathbf{a}{\restriction}_{V_{(i,\alpha)}}$. □

KEY: Each $\mathit{Inst}^B_{(i,\alpha)}$ can be solved by Gaussian elimination (!), and there are only poly(n)-many of them. Hence (using the above Lemma) we can "easily" pre-process *Inst* to enforce the following:

For every prime p, $(i,\alpha) \in VC_p$, $s \subseteq V_{(i,\alpha)}$ with $|s| = 2$, and $\mathbf{c} \in C_s$, there exists $\mathbf{b} \in \mathrm{Sol}(\mathit{Inst}_{(i,\alpha)})$ with $\mathrm{proj}_s(\mathbf{b}) = \mathbf{c}$.

Call this condition **2-linear consistency**.

APOLOGY: there is one more technical definition.

It takes 3 slides to explain.

Active 3-ary constraints

Again assume *Inst* is a 3-minimal instance of 3-CSP($\mathbf{A}$).

For any $|s| = 3$ we always have

$$C_s \;\subseteq\; \underbrace{\{\mathbf{a} \in A^s \,:\, \mathrm{proj}_t(\mathbf{a}) \in C_t \text{ for all } t \subseteq s \text{ with } |t| = 2\}}_{\widehat{C_s}}.$$

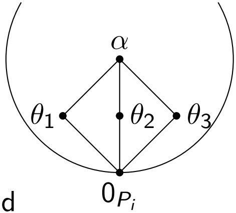
Definition. Call (s, C_s) **passive** if $C_s = \widehat{C_s}$, and **active** if $C_s \subsetneq \widehat{C_s}$.

(Aside: if we start with an instance of 2-CSP($\mathbf{A}$) and enforce 3-minimality, all of the resulting 3-ary constraints will be passive.)

Example – active constraint

Suppose $i \in V$, $\alpha \in \mathrm{Con}(\mathbf{P}_i)$, and $\langle 0_{P_i}, \alpha \rangle$ bounds a copy of $\mathbf{M}_3$.

$\mathrm{Con}(\mathbf{P}_i) =$



Suppose also that $j, k, \ell \in V$ and

$$
\begin{aligned}
\mathbf{P}_j \ \text{“=”}\ \mathbf{P}_i/\theta_1 &\qquad \text{via } h_{ij} : \mathbf{P}_i \to \mathbf{P}_j \\
\mathbf{P}_k \ \text{“=”}\ \mathbf{P}_i/\theta_2 &\qquad \text{via } h_{ik} : \mathbf{P}_i \to \mathbf{P}_k \\
\mathbf{P}_\ell \ \text{“=”}\ \mathbf{P}_i/\theta_3 &\qquad \text{via } h_{i\ell} : \mathbf{P}_i \to \mathbf{P}_\ell
\end{aligned}
$$

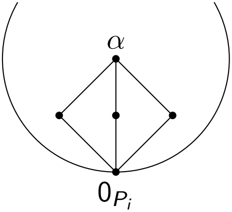
Define

$$H \ = \ \{(h_{ij}(a), h_{ik}(a), h_{i\ell}(a)) \ : \ a \in P_i\} \ \subseteq \ A^{\{j,k,\ell\}}.$$

Let $s = \{j, k, \ell\}$. Then $H \subsetneq \widehat{C_s}$. Hence if $C_s = H$, then (s, C_s) is active.

$\mathbf{M}_3$-induced active constraints

$\mathrm{Con}(\mathbf{P}_i) =$



Definition. Let **A** be a finite ternary group and *Inst* a 3-minimal instance of 3-CSP(**A**). We say that *Inst* has **$\mathbf{M}_3$-induced active constraints** if for every $i \in V$, $\alpha \in \mathrm{Con}(\mathbf{P}_i)$, and $s = \{j, k, \ell\} \subseteq V$ as described on the previous slide, and with $H = \{(h_{ij}(a), h_{ik}(a), h_{i\ell}(a)) : a \in P_i\}$,

if α is elementary p-abelian for some prime p, **then** $C_s = H$.

By adding the constraint (s, H) whenever required, we can easily enforce that *Inst* have $\mathbf{M}_3$-induced active constraints.

Pre-processing: Summary

Let $\mathbf{A}$ be a finite ternary group. Given an instance of 2-CSP($\mathbf{A}$), we can enforce

- 3-minimality
- Congruence completeness.
- 2-linear consistency
- $\mathbf{M}_3$-induced active constraints.

If a contradiction is not found, this "super" pre-processing will produce an equivalent instance of 3-CSP($\mathbf{A}$) which:

- is 3-minimal;
- is congruence complete;
- is 2-linearly consistent;
- has $\mathbf{M}_3$-induced active constraints;
- has no other active constraints.

Conjecture (Stará Lesná)

Suppose **A** is a finite ternary group and *Inst* is an instance of 3-CSP(**A**) which:

- is 3-minimal;
- is congruence complete;
- is 2-linearly consistent;
- has $\mathbf{M}_3$-induced active constraints;
- has no other active constraints.

Then *Inst* has a solution.

If true, we will get the following "easy" algorithm for ternary groups **A**:

Input: an instance of 2-CSP(**A**)
"Super" pre-process the instance
    If a contradiction is found, return "NO"
Return "Yes"

OK, **maybe** this new algorithm will work for ternary groups . . .

. . . but what does this have to do with Taylor algebras??

Generalizing to Taylor algebras

For general algebras, there is a notion of "abelian congruence."

If **A** is finite and has a Taylor term, then every block of an abelian congruence "is" a ternary abelian group (in a natural way).

Definition. Let **A** be a finite algebra with a Taylor term, $\alpha \in \mathrm{Con}(\mathbf{A})$, and *p* a prime. We say that α is **elementary *p*-abelian** if α is abelian and every α-block "is" a ternary abelian group of exponent *p*.

Many facts about finite ternary groups lift to abelian congruences in finite Taylor algebras. For example:

Proposition

Let **A** be a finite algebra with a Taylor term and α a minimal congruence. If α is abelian, then α is elementary *p*-abelian for some prime *p*.

Wild speculation

Let **A** be any finite, idempotent algebra with a Taylor term.

Let *Inst* be an instance of 2-CSP(**A**).

Just as for ternary groups, we can "super" pre-process *Inst* to either find a contradiction or produce an equivalent instance of 3-CSP(**A**) which:

- is 3-minimal;
- is congruence complete;
- is 2-linearly consistent;
- has $\mathbf{M}_3$-induced active constraints;
- has no other active constraints.

Problem (Stará Lesná)

For which Taylor varieties is it true that every 3-CSP(**A**) instance satisfying the above conditions has a solution? (Could it be <u>all</u> Taylor varieties??)

Thank you!