Ideals and involutive filters in residuated lattices

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INVESTMENTS IN EDUCATION DEVELOPMENT

The class of bounded integral residuated lattices contains some classes of algebras behind many-valued and fuzzy logics, e.g.:

MV-algebras - Łukasiewicz infinite valued logic

BL-algebras - Hájek's basic (fuzzy) logic

MTL-algebras - monoidal t-norm based logic

Classes of non-commutative variants (pseudo MV-algebras = GMV-algebras, pseudo BL-algebras, pseudo MTL-algebras).

Heyting algebras - intuitionistic logic

A bounded integral residuated lattice is an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0, \rangle$ satisfying the following conditions:

(i) $(M; \odot, 1)$ is a monoid;

(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice;

(iii) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$ for any $x, y \in M$.

residuated lattice = bounded integral residuated lattice.

If the operation $\odot$ on a residuated lattice M is commutative then M is called commutative residuated lattice. In such a case the operations $\rightarrow$ and $\rightsquigarrow$ coincide. In a residuated lattice M we define two unary operations (negations) $^{-}$ and $^{\sim}$ as follows: $x^{-} = x \rightarrow 0$, $x^{\sim} = x \rightsquigarrow 0$ for each $x \in M$.

A residuated lattice M is

- a pseudo MTL-algebra if M satisfies the identities of pre-linearity
  (iv) $(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x)$;
- an $R\ell$-monoid if M satisfies the identities of divisibility
  (v) $(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$;
- a pseudo BL-algebra if M satisfies both (iv) and (v);
- involutive if M satisfies the identities
  (vi) $x^{-\sim} = x = x^{\sim -}$;
- a GMV-algebra (or equivalently a pseudo MV-algebra) if M satisfies (iv), (v) and (vi);
- a Heyting algebra if the operations $\odot$ and $\wedge$ coincide.

M is called good, if it satisfies the identity $x^{-\sim} = x^{\sim -}$.

M is called normal if it satisfies the identities $(x \odot y)^{-\sim} = x^{-\sim} \odot y^{-\sim}$ and $(x \odot y)^{\sim -} = x^{\sim -} \odot y^{\sim -}$.

A non-empty subset F of a residuated lattice M is called a filter of M if

(F1) $x, y \in F$ imply $x \odot y \in F$;

(F2) $x \in F,\ y \in M,\ x \leq y$ imply $y \in F$.

A filter F is called normal if

(F3) $x \to y \in F \iff x \rightsquigarrow y \in F,\ \forall x, y \in M.$

normal filters of M $\longleftrightarrow$ kernels of congruences on M

$\langle x, y \rangle \in \theta_F \iff (x \to y) \wedge (y \to x) \in F;$

$\langle x, y \rangle \in \theta_F \iff (x \rightsquigarrow y) \wedge (y \rightsquigarrow x) \in F.$

The corresponding quotient residuated lattice: $M/\theta_F = M/F$.

In GMV-algebras: the operations multiplication and addition are mutually dual, hence there exists the notion dual to a filter, i.e. an ideal. The theories of filters and of ideals in GMV-algebras are so mutually dual.

In general residuated lattices: a dual operation to the multiplication does not exist. Then a notion of "the (precise) dual to filter" does not exist too.

Nevertheless, we can introduce some kind of an ideal (not dual to a filter) in any residuated lattice which is useful in the study of structure of residuated lattices.

Let M be a residuated lattice. For any $x, y \in M$ we put

$$x \oslash y := y^{-} \rightsquigarrow x, \quad x \obslash y := x^{\sim} \to y.$$

The operation $\oslash$ will be called left addition and $\obslash$ will be called right addition on M.

Proposition

If M is a good and normal residuated lattice, then both left and right additions are associative.

Let M be any residuated lattice. A non-empty subset I of M is called a left ideal of M if:

1. $x, y \in I \implies x \oslash y \in I$;
2. $x \in I,\ z \in M,\ z \leq x \implies z \in I$.

Theorem

Let I be a subset of a residuated lattice M containing 0. Then I is a left ideal of M if and only if

$$\forall x, y \in M;\ x^{-} \odot y \in I, x \in I \implies y \in I. \qquad (*)$$

A non-empty subset I of a residuated lattice M is called a right ideal of M if:

1'. $x, y \in I \implies x \oslash y \in I$;

2. $x \in I,\ z \in M,\ z \leq x \implies z \in I$.

Theorem

Let I be a subset of a residuated lattice M containing 0. Then I is a right ideal of M if and only if

$$\forall x, y \in M;\ y \odot x^{\sim} \in I, x \in I \implies y \in I. \qquad (**)$$

Every left ideal as well as every right ideal of a residuated lattice M is a lattice ideal.

Let M be a residuated lattice and $\emptyset \neq I \subseteq M$. Then I is called an ideal of M if it is both left and right ideal of M.

Theorem

If $\emptyset \neq I \subseteq M$ then I is an ideal of M if and only if

1. $x, y \in I \implies x \oslash y \in I$;

1'. $x, y \in I \implies x \otimes y \in I$;

2. $x \in I,\ z \in M,\ z \leq x \implies z \in I$.

If I is an ideal of a residuated lattice M, define the binary relation θ_I on M as follows ($x,\ y \in M$):

$$\langle x,\ y\rangle \in \theta_I \ :\Longleftrightarrow\ x^- \odot y \in I,\ y^- \odot x \in I,\ x \odot y^\sim \in I,\ y \odot x^\sim \in I.$$

Theorem

If M is a residuated lattice and I is an ideal of M, then θ_I is an equivalence on M which is a congruence on the reduct $(M;\ \odot, \vee, \rightarrow, \rightsquigarrow, 0, 1)$ of the residuated lattice M.
If M is a pseudo BL-algebra then θ_I is a congruence on M.

Theorem

a) If M is a pseudo BL-algebra and I is an ideal of M, then M/θ_I is a GMV-algebra.
b) If M is any residuated lattice then M/θ_I is an involutive residuated lattice.

If F is a normal filter of a residuated lattice M, then we say that F is an involutive filter if the quotient residuated lattice M/F is involutive.

Let M be a residuated lattice. Then we say that M satisfies the Glivenko property if for any $x, y \in M$

$$(x \to y)^{-\sim} = x \to y^{-\sim}, \ \ (x \rightsquigarrow y)^{\sim -} = x \rightsquigarrow y^{\sim -}. \qquad \text{(GP)}$$

If a residuated lattice M is good, then every of the conditions

(i) $(x^{-\sim} \to x)^{-\sim} = 1 = (x^{\sim -} \rightsquigarrow x)^{\sim -}$ for every $x \in M$,

(ii) $(x \to y)^{-\sim} = x^{-\sim} \to y^{-\sim}$, $(x \rightsquigarrow y)^{\sim -} = x^{\sim -} \rightsquigarrow y^{\sim -}$, for every x, y

is equivalent to (GP).

For example, every good $R\ell$-monoid satisfies (GP).

If M is a residuated lattice then we denote
$D(M) := \{x \in M : x^{-\sim} = 1 = x^{\sim -}\}$, the set of dense elements of M.

Theorem

(i) If M is a good residuated lattice, then $D(M)$ is a filter of M.
(ii) If, moreover, M satisfies (GP), then $D(M)$ is a normal filter of M.

Theorem

Let M be a good residuated lattice satisfying (GP) and $x, y \in M$. Then $\langle x, y \rangle \in \theta_{D(M)}$ if and only if $x^{-\sim} = y^{-\sim}$. Moreover, $M/D(M)$ is an involutive residuated lattice, i.e. $D(M)$ is an involutive filter.

An element x of a residuated lattice M is called regular if $x^{-\sim} = x = x^{\sim -}$. Denote by $Reg(M)$ the set of all regular elements in M.

Theorem

If M is a good pseudo BL-algebra, then $Reg(M)$ is a subalgebra of M and $Reg(M)$ is isomorphic to $M/D(M)$.

Theorem

If a good residuated lattice M satisfies (GP) and F is an involutive normal filter of M, then $D(M) \subseteq F$.

Theorem

If M is a good residuated lattice satisfying (GP) then the involutive filters of M are exactly all normal filters of M containing $D(M)$.

Proposition

If M is a residuated lattice and I is an ideal of M then I is the 0-class in M/θ_I.

Proposition

Let I be an ideal of a pseudo BL-algebra and $F = F_I = 1/\theta_I$. Then F is an involutive normal filter of M.

Proposition

If M is a residuated lattice and F is a normal filter of M, then the class $0/F$ is an ideal of M.

An algebra A is congruence regular if each its congruence is determined by any one of its congruence class, i.e. if $x/\theta = x/\phi$ for some $x \in A$ then $\theta = \phi$. For algebras with a constant 1 a weaker version of congruence regularity is 1-regularity, i.e. for all congruence θ, ϕ on A, $1/\theta = 1/\phi$ implies $\theta = \phi$.
Residuated lattices are 1-regular, but not regular. Hence every congruence θ on a residuated lattice M is determined uniquely by the filter $F_\theta = 1/\theta$, but other classes in M/θ can be at the same time also classes in different congruences on M.

Nevertheless we have:

Theorem

If M is an arbitrary pseudo BL-algebra then there is a one-to-one correspondence between ideals and involutive normal filters of M.

Example.

$M = \{0, a, 1\}$ ... Heyting algebra, $0 < a < 1$.

$\odot = \wedge \Longrightarrow$ filters of M = lattice filters.

Filters $F_1 = \{a, 1\}$, $F_2 = \{1\}$, corresponding congruences θ_1, θ_2,
$0/\theta_1 = 0/\theta_2 = \{0\}$.

M/θ_1 is an involutive residuated lattice, while M/θ_2 is not involutive.

M has exactly filters: F_1, F_2 and $F_3 = \{0, a, 1\} = M$.

$D(M) = \{a, 1\}$, hence $D(M) \subseteq F$ iff $F = F_1$ or $F = F_3$.

The ideals of M : $\{0\}$ and M.

(The lattice ideal $\{0, a\}$ is not an ideal of M.)