# Unitizing generalized pseudo effect algebras

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# Introduction

Effect algebras (EAs) $(E; \oplus, 0, 1)$ (D.J. Foulis and M.K. Bennett, 1994) — abstract version of the system of all self-adjoint operators between zero and identity on a Hilbert space – algebraic basis for (possibly) unsharp (fuzzy) quantum-mechanical measurements.

Generalized effect algebras (GEAs) $(E; \oplus, 0)$ – EAs with no largest element — continue the study of generalized Boolean algebras (M.H. Stone, 1935) and generalized orthomodular lattices (M.F. Janowitz, 1968).

A noncommutative version of EAs — *pseudo effect algebras* (PEAs) and *generalized pseudo effect algebras* (GPEAs) (again with no largest element) were introduced and studied by A. Dvurečenskij and T. Vetterlein, 2001.

# outline

Every GEA $E$ can be embedded as a maximal EA-ideal in an EA $\hat{E}$ — *unitization* of $E$, such that (1) $\forall a \in \hat{E}$, either $a \in E$ or $a^{\perp} = 1 \ominus a \in E$, (2) $\forall x, y \in \hat{E} \setminus E$, $x \oplus y$ in $\hat{E}$ is undefined, (3) $E$ and $\hat{E} \setminus E$ are order-anti-isomorphic ( Hedlíková and SP, 1996, Wilce, 1998).

We show that, analogously, a GPEA $P$ can be embedded as a maximal normal PEA-ideal in a PEA $U$ if and only if $P$ admits a *unitizing GPEA-automorphism*. For a special case of *weakly commutative* GPEAs such embedding was accomplished by Xie and Li, 2010, by using the identity mapping as the unitizing GPEA-automorphism.

We also characterize PEAs that arise as unitizations of GPEAs as those PEAs that admit a two-valued state.

# pseudo effect algebras

A *pseudo effect algebra* (PE) is a structure $(P; \oplus, 0, 1)$, where $\oplus$ is a partial binary operation on $E$, such that:

(PE1) $\oplus$ is associative in the sense that $(a \oplus b) \oplus c$ is defined iff $a \oplus (b \oplus c)$ is defined, and in this case $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (*associativity*).

(PE2) If $a \oplus b$ is defined then $a \oplus b = c \oplus a = b \oplus d$ for some $c, d \in P$ (*conjugation*).

(PE3) For every $a \in E$ there exist unique $a^-, a^\sim \in E$ such that $a^- \oplus a = 1 = a \oplus a^\sim$ (*supplementation*).

(PE4) If $a \oplus 1$ or $1 \oplus a$ is defined then $a = 0$ $(0, 1$*law*).

# generalized PEA

A *generalized pseudo effect algebra* (GPEA) is a structure $(P; \oplus, 0)$, where $\oplus$ is a partial binary operation on $E$, such that:

(GPEA1) $(a \oplus b)$ and $(a \oplus b) \oplus c$ exist iff $b \oplus c$ and $a \oplus (b \oplus c)$ exist and in this case $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (*associativity*).

(GPEA 2) If $a \oplus b$ exists, then there are $c, d \in P$ such that $a \oplus b = c \oplus a = b \oplus d$ (*conjugation*).

(GPEA3) $a \oplus 0 = 0 \oplus a = a$ (*neutral element*).

(GPEA4) If $a \oplus c = b \oplus c$, or $c \oplus a = c \oplus b$, then $a = b$ (*cancellation*).

(GPEA5) If $a \oplus b = 0$, then $a = b = 0$ (*positivity*).

# partial order, subtractions

On a PEA $(P; \oplus, 0, 1)$ (GPEA $(P; \oplus, 0)$) a partial order $\leq$ is defined by

$$a \leq b \text{ iff } b = x \oplus a \text{ for some } x \in P$$

$$\text{iff } b = a \oplus y \text{ for some } y \in P$$

In this ordering, $0 \leq a \leq 1$ for all $a \in P$.

Partial subtractions $\diagdown, \diagup$ are defined by

$$b \diagdown a = x \text{ iff } b = x \oplus a;$$

$$a \diagup b = y \text{ iff } b = a \oplus y.$$

Then (in a PEA), $a^- = 1 \diagdown a, \; a^\sim = a \diagup 1$.

# basic properties of PEAs

- $a^{-\sim} = a = a^{\sim -}$; $1^{-} = 0 = 1^{\sim}$; $a \oplus 0 = a = 0 \oplus a$.
- $a \oplus b$ is defined iff $a \leq b^{-}$ iff $a^{\sim} \geq b$.
- $a \oplus b = c$ iff $a^{\sim} = b \oplus c^{\sim}$ iff $b^{-} = c^{-} \oplus a$.
- If $a \leq b$, then $b \diagdown a = (a \oplus b^{\sim})^{-}$ and $a \diagup b = (b^{-} \oplus a)^{\sim}$.
- Both $a \mapsto a^{\sim}$ and $a \mapsto a^{-}$ are order-reversing bijections on $P$.

# examples

Example 1. Let $(G; +, 0, \leq)$ be a po-group, $u \in G, u \geq 0$. Then $G[0, u] := \{x \in G : 0 \leq x \leq u\}$ with $+$ restricted to $a, b \in G[0, u]$: $a \oplus b = a + b$ iff $a + b \leq u$, is a PEA.

Example 2. $G^+ := \{a \in G : a \geq 0\}$ with $a \oplus b = a + b, a, b \in G^+$, is a GPEA (here $\oplus$ is totally defined).

Example 3. $0 \in S$ – a convex subset of $G^+$, define $a \oplus_S b := a + b$ iff $a + b \in S$, $(S; \oplus_S, 0)$ is a GPEA.

# morphisms

- $P, Q$ — GPEAs, $\phi : P \to Q$ is a *GPEA-morphism* iff $\forall a, b \in P$, $\exists a \oplus b \implies \exists\, \phi(a) \oplus \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$.
- If $\phi : P \to Q$ is a bijective GPEA-morphism, $\phi^{-1} : Q \to P$ is also a GPEA-morphism, then $\phi$ is a *GPEA-isomorphism* of $P$ onto $Q$. A GPEA-isomorphism $\phi : P \to P$ is a *GPEA-automorphism* of $P$.
- $P, Q$ — PEAs, $\phi : P \to Q$ is a *PEA-morphism* iff it is a GPEA-morphism and $\phi(1) = 1$.
- If $\phi : P \to Q$ is a bijective PEA-morphism, and $\phi^{-1}$ is also a PEA-morphism, then $\phi$ is a *PEA-isomorphism* of $P$ onto $Q$. A PEA-morphism $\phi : P \to P$ is a *PEA-automorphism* of $P$.

# special automorphisms

$P$ — PEA, then

- The mappings $a \mapsto a^{--}$ and $a \mapsto a^{\sim\sim}$ are PEA-automorphisms of $P$ and they are inverse of each other.
- $\phi : P \to P$. TFAE:
  - $\forall a, b \in P$, $\phi(a) \oplus b$ exists in $P \Leftrightarrow b \oplus a$ exists in $P$.
  - $\forall a \in P$, $(\phi(a))^{\sim} = a^{-}$.
  - $\phi(a) = a^{--}\ \forall a \in P$.

# intervals,subalgebras, ideals

$P$ — GPEA

• $\forall a \in P$, $P[0,a] := \{x \in P : x \leq a\}$ can be made a PEA defining $x \oplus_a y = x \oplus y$ whenever $x \oplus y \leq a$.

• $\emptyset \neq S \subseteq P$, $S$ is a *sub-GPEA* of $P$ iff $\forall a, b, c \in P$, $a \oplus b = c$, if two of $\{a, b, c\}$ belong to $S$, then the third also belongs to $S$.

• $\emptyset \neq I \subseteq P$ is an *ideal* iff
(Ii) $b \in I, a \in P, a \leq b \implies a \in I$;
(Iii) $a, b \in I, a \oplus b \in P \implies a \oplus b \in I$.
An ideal $I$ is *normal* iff
(N) $a, b, c \in P, a \oplus c = c \oplus b \implies a \in I \Leftrightarrow b \in I$.

# unitization of a GPEA

- A PEA $(U;+,0,1)$ is a *binary unitization* of a GPEA $(P;\oplus,0)$ iff:

(UGPEA1) $P \subseteq U$,
$a \oplus b$ exists in $P \Leftrightarrow a+b$ exists in $U$,
and $a \oplus b = a+b$.

(UGPEA2) $1 = \not\in P$

(UGPEA3) $x, y \in U \setminus P \implies x+y$ is undefined.

- A GPEA-automorphism $\gamma : P \to P$ is said to be *unitizing* iff $\forall a, b \in P$, $\gamma a \oplus b$ is defined $\Leftrightarrow b \oplus a$ is defined.

# construction

- $(P; \oplus, 0)$ — GPEA, $\gamma : P \to P$ — unitizing GPEA-automorphism,
$P^\eta$ — a set disjoint from $P$ with the same cardinality,
$\eta : P \to P^\eta$ — bijection.
$U := P \cup P^\eta$, $+$ — partial binary operation on $U$:
(1) $a, b \in P$, $a + b$ is defined iff $a \oplus b$ is defined and $a + b = a \oplus b$.
(2) $a, b \in P$, $x \in U \setminus P$, $x = \eta b$, $a + \eta b$ is defined iff $a \leq b$ and $a + \eta b := \eta(b \diagdown a)$.
(3) $a, b \in P$, $y \in U \setminus P$, $y = \eta a$, $\eta a + b$ is defined iff $\gamma b \leq a$, $\eta a + b := \eta(\gamma b \diagup a)$.
(4) $a, b \in U \setminus P$, $x + y$ is undefined.
Then, with $1 := \eta 0$, $(U; +, 0, 1)$ is a PEA — the $\gamma$- *unitization* of $P$.

# results

• A necessary and sufficient condition for a GPEA $P$ to admit a unitization $U$ is that it admits a unitizing GPEA-automorphism.

• $P$ is a maximal normal ideal in $U$.
A *state* on a PEA $P$ is a mapping $s : P \to [0, 1] \subset \mathbb{R}$ such that (1) $s(1) = 1$, (2) $s(a + b) = s(a) + s(b)$ whenever $a + b$ exists in $P$. A state $s$ is *two-valued* iff $\forall a \in P, s(a) \in \{0, 1\}$.
• $(U; +, 0, 1)$ — a PEA. There exists a a maximal normal ideal $I$ in $U$ such that $(U; +, 0, 1)$ is a unitization of $(I; \oplus, 0)$, where $\oplus$ is the restriction of $+$ to $I$, iff $P$ admits a two-valued state. In fact, $I = \{a \in U : s(a) = 0\}$.

# weakly commutative GPEAs

- A GPEA $P$ is *weakly commutative* iff $\forall a, b \in P$, $a \oplus b$ exists $\Leftrightarrow b \oplus a$ exists.
- A PEA $P$ is *symmetric* iff $\forall a \in P$, $a^{-} = a^{\sim}$.
- A PEA $P$ is symmetric iff it is weakly commutative.

• Identity mapping is a unitizing automorphism in a GPEA $P$ iff $P$ is weakly commutative.

• If in a GPEA $P$, $a \oplus b$ is defined for all $a, b \in P$, then every automorphism of $P$ is unitizing.

• In a PEA $P$, the unique unitizing automorphism is $\gamma a = a^{--}$.

# example

$G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$,

$$(a,b,c) + (x,y,z) = \begin{cases} (a+x, b+y, c+z) & : \quad x \text{ is even} \\ (a+x, c+y, b+z) & : \quad x \text{ is odd} \end{cases}$$

$(a,b,c) \le (x,y,z)$ iff $a < x$ or $a = x, b \le y, c \le z$.

$(G, +, \le)$ — lattice-ordered group with strong unit $u = (1,0,0)$,

$E := G[(0,0,0),(1,0,0)]$ is a PEA.

Elements between $(0,0,0)$ and $(1,0,0)$ are of two kinds—$(0,a,b)$ and $(1,c,d)$—where $a,b \ge 0$ and $c,d \le 0$.

$P := \{(0,a,b) \in G : a,b \ge 0\}$ — maximal normal ideal of $E$,

$\gamma(0,a,b) = (0,a,b)^{--} = (0,b,a)$ — unitizing automorphism,

$E$ — $\gamma$-unitization of $P$.

# kite construction

Dvurečenskij, 2013: a GPEA $P$, $\emptyset \neq I$, bijections $\lambda, \rho : I \to I$, on $P^I$:

(C1) $\exists a_{\rho i} \oplus b_i \Leftrightarrow \exists b_i \oplus a_{\lambda i}$, (C2) $\exists a_{\lambda i} \oplus b_i \Leftrightarrow \exists b_i \oplus a_{\rho i}$.

Put $K := P^I \dot{\cup} (P^I)^\eta$ and define:

(K1) $(a_i)_{i \in I} + (b_i)_{i \in I} := (a_i \oplus b_i)_{i \in I}$ iff $a_i \oplus b_i$ exists in $P$ for all $i \in I$.

(K2) $(a_i)_{i \in I} + (\eta b_i)_{i \in I} := (\eta(b_i \diagdown a_{\lambda i}))_{i \in I}$ iff $a_{\lambda i} \leq b_i$ for all $i \in I$.

(K3) $(\eta a_i)_{i \in I} + (b_i)_{i \in I} := (\eta(b_{\rho i} \diagup a_i))_{i \in I}$ iff $b_{\rho i} \leq a_i$ for all $i \in I$.

(K4) $(\eta a_i)_{i \in I} + (\eta b_i)_{i \in I}$ is undefined.

Then $K$ is the $\gamma$-unitization of $P^I$, $\gamma(a_i) = (a_{\rho \circ \lambda^{-1} i})$.

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