Algebraic Approach to Image Segmentation

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1 Geometric spaces
- Generalized (ultra)metric
- Segmentational spaces
- Generalized normed affine spaces

2 Image segmentation
- Digital images
- Segmentation
- Universal method
- General and algebraic methods
- General and algebraic methods

Generalized metric

Ordered algebra: each operation is isotone in each component

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Let P be a set of points and $(S, +, 0, \leq)$ be an ordered monoid with the least element 0. A mapping $\delta : P^2 \to S$ will be called a generalized metric *(GM) if*

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convex set (via Mazur intersection property) intersection of closed balls; they form a set $\mathcal{M}_\delta$

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Ultrametric = stronger metric, the triangle inequality is replaced by a stronger property.

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Let P be a set of points and $(S, \vee)$ be a join semilattice. A mapping $\delta : P^2 \to S$ will be called a generalized ultrametric *(GUM) if it satisfies strong self-uniformity, symmetry and*

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The triple (P, S, δ) is called a generalized metric space *(a GUM space).*

Clearly each GUM is a GM.

Properties of GUM spaces

Let (P, S, δ) be a GUM space and $(S, \leq)$ be a complete lattice.

Proposition

- *Every element of a ball is its center.*
- *Nonempty intersection of two balls is a balls (with the meet of radii as the radius).*
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The smallest radius of a ball B will be called *the norm* of B and denoted by $n(B)$.

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Example

The collection of closed bounded convex sets in the Euclidean space $\mathbb{R}^n$ for any $n > 0$ forms a segmentational space.

Relation between GUM and segmentation spaces

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$$\lambda(X) = \{X\} \cup \{\{a\} \mid a \in A \setminus X\} \tag{1}$$

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Thus each segmentational space is a GUM space.

Generalized normed affine spaces

A *generalized normed affine space* (abbreviated as *GNAS*) is a 3-sorted structure consisting of a triple (P, V, S) of sets of *points*, *vectors* and *scalars*, with elements typically denoted by $x, y, z \in P$, $u, v, w \in V$ and $a, b, c \in S$ (used bellow) with the structure:

1. an ordered abelian semigroup $(V, +, \leq)$, an ordered commutative unital semiring $(S, \oplus, \odot, 0, 1, \leq)$ with $0 \leq 1$
2. a *subaction* $\bullet : S \times V \to V$ (with the computational priority to addition) satisfying isotonicity in the second argument and :
   1. $a \bullet (b \bullet v) \leq (a \odot b) \bullet v$,
   2. $a \bullet v + a \bullet w \leq a \bullet (v + w)$,
   3. $a \bullet v + b \bullet v \leq (a \oplus b) \bullet v$
   4. $1 \bullet v = v$
3. a *segment function* $\star : P \times P \to V$ with the triangle inequality $x \star z \leq x \star y + y \star z$,

GNAS - continued

6. an isotone *subnorm* $\nu : V \to S$ satisfying the following axioms:
   1. the elements greater or equal to $\nu(u)$ for some u are called *non-negative* and form a submonoid $S_{\geq 0}$ of $(S, \oplus)$
   2. $(P, S_{\geq 0}, \delta)$, where $\delta(x, y) = \nu(x \star y)$, is a generalized metric space.
7. We say a pair of vectors v, w is colinear (we write $v \sim w$) iff there exists a unit vector u (i.e., $\nu(u) = 1$) such that $v = \nu(v) \bullet u$ and $w = \nu(w) \bullet u$. In such a case we require:
   1. $\nu(v + w) = \nu(v) + \nu(w)$,
   2. $\nu(\nu(v) \bullet w) \leq \nu(v) \odot \nu(w)$.

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- Each GUM space has a structure of GNAS. The convex sets, i.e., the closed balls, play the role of vectors.
- $(\mathbb{R}^n, \mathcal{M}_\epsilon, \mathbb{R}_0^+)$ with $\mathcal{M}_\epsilon$ being the set of nonempty bounded closed convex sets forms a GNAS. The subaction is given by an *a*-hull and the subnorm is a diameter of the set.

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A digital image *is a triple* (P, π, f) *where* (P, π) *is a finite digital space and* $f : P \to (S, \leq)$ *is an assignment of* colors. $(S, \leq)$ *is a poset with a greatest element* $\top$*.*

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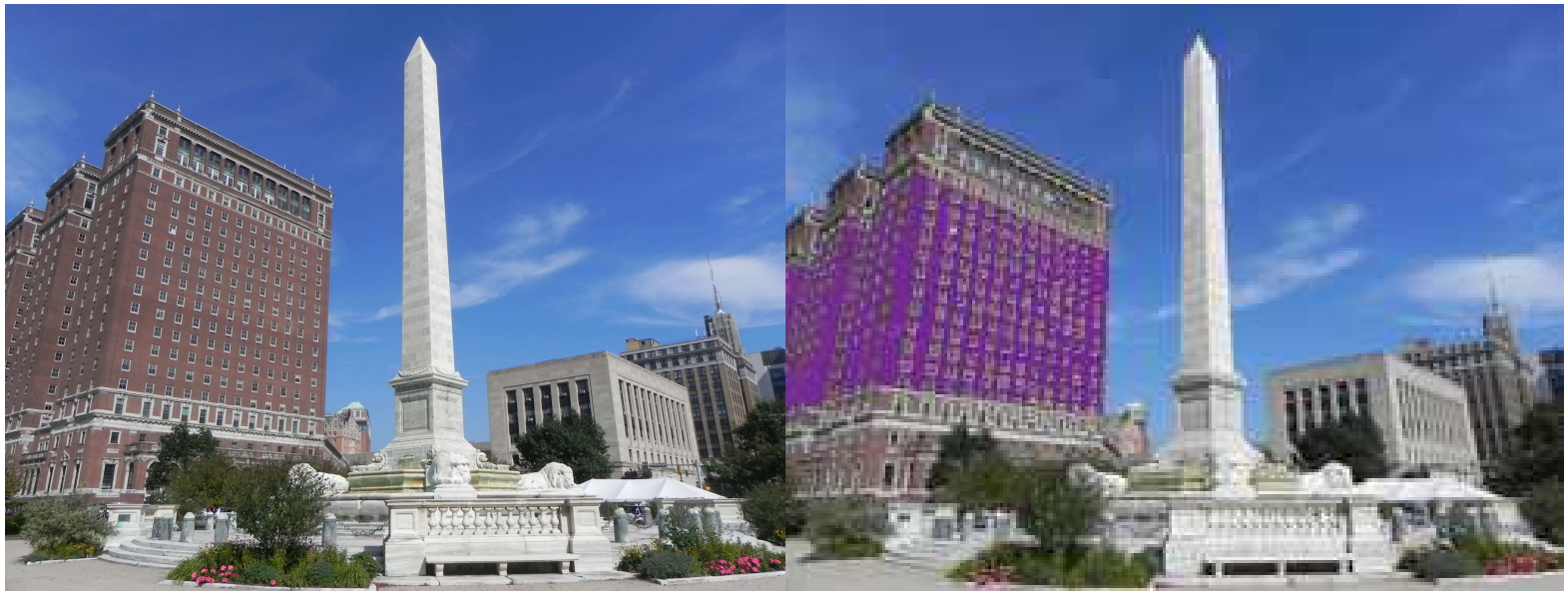
A digital image can be seen as a distribution of a certain physical quantity over a digitized real of virtual space.

Segmentation of a digital image

Image segmentation is a decomposition of the image into meaningful parts (*admissible sets*) which correspond to the objects in the original space.



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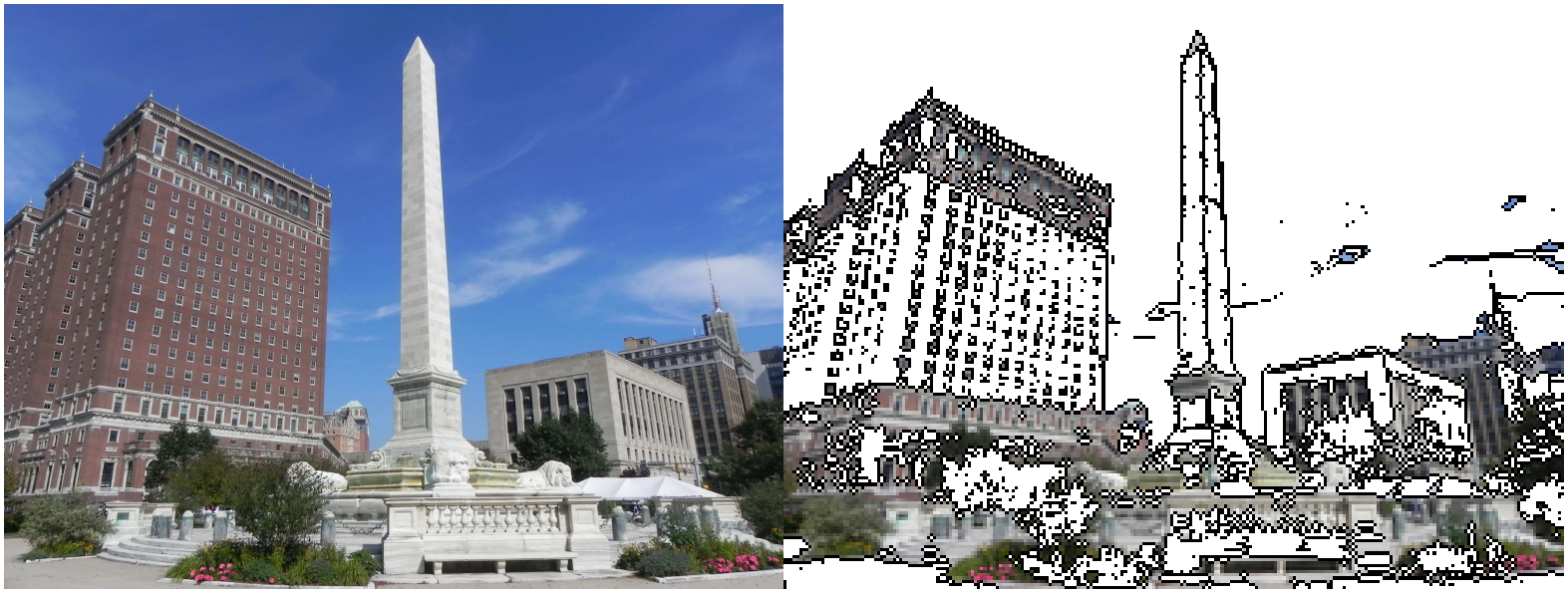

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There are plenty of kinds of segmentations - based on both global properties (e.g., from the histogram, watersheds, region divisions) and local ones (e.g. fuzzy segmentation, region growing methods, merging the areas, etc.) and others.

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In general, if we assume that the admissible sets gives rise to a segmenatational space on the set of pixels. The admissibility must be derived from the image itself. One way to do it is via a criterion of similarity.

Similarity criterion

Let P be a finite set of pixels. A collection of quantities determining inclusion of a pixel into an object can be merged into a single mapping – *a criterion* $\xi : P^2 \to (S, \leq)$ to some (finite) poset.

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Example

Consider the graph distance d in the digital space and 3 channels R,G,B as functions $P \to \{0, \ldots, 255\}$ (referred to as *initial quantities*). Then we can take as a criterion

$$\xi(x, y) = (d(x, y), |R(x) - R(y)|, |G(x) - G(y)|, |B(x) - B(y)|).$$

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Since the order on $\mathbb{R}^4$ is not linear, the question arises, how to treat the criterion to get admissible sets corresponding to such situation.

L-fuzzy method

We start with a finite set P and a criterion $\xi : P^2 \to (S, \leq)$, satisfying $\xi(x, x) = \top$ and symmetry. We want to find corresponding admissible sets and segmentations.

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- each set $\theta(x, \tau)$ for $x \in P$ is closed under collective similarity on the level τ,
- nice segmentations are namely those given by fixed threshold term τ (*algebraic segmentations*).

Universal method

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Definition

Let $\mathcal{I}$ be a digital image described by:

- *a set P of pixels,*
- *a lattice L of levels of similarity w.r.t. the image,*
- *a similarity $\kappa : P^2 \to L$ (a criterion of pixel similarity).*

Then algebraic segmentations (for some threshold $t \in L$) are partitions into the sets of the form $\{b \in P \mid \kappa(a, b) \geq t\}$ with $a \in P$.

The method (for a given image) is called *algebraic*, if each segmentation is algebraic.

The properties of the general method

Proposition

1. *The admissible sets are balls w.r.t. a certain generalized ultrametric.*
2. *The image $\mathcal{I}$ can be treated as a GNAS where*
   - *points = pixels,*
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Theorem

- *Each parametric segmentational method with the intersection property is an instance of the algebraic method.*
- *Each segmentational method with the distributive target lattice is an instance of a L-fuzzy segmentation for a certain criterion.*

Acknowledgement

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Thank you for your attention.