Residuated structures and certain triples

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Residuated structures

1. Residuated (integral commutative) ℓ-monoids (aka residuated lattices) are algebras $\langle A; \vee, \wedge, \cdot, \to, 1\rangle$ where $\langle A; \vee, \wedge, 1\rangle$ is a lattice with 1, $\langle A; \cdot, 1\rangle$ is a commutative monoid, and $\forall x, y, z \in A$:

$$z \leq x \to y \;\Leftrightarrow\; z \cdot x \leq y. \tag{Res}$$

2. Residuated (integral commutative) po-monoids are structures $\langle A; \leq, \cdot, \to, 1\rangle$ where $\langle A; \leq, 1\rangle$ is a poset with 1, $\langle A; \cdot, 1\rangle$ is a commutative monoid, and (Res) holds.

3. BCK-algebras = subalgebras of the $\{\to, 1\}$-reducts of residuated ℓ-monoids = algebras $\langle A; \to, 1\rangle$ of type $\langle 2, 0\rangle$ satisfying

$$1 \to x = x, \quad x \to 1 = 1,$$

$$(x \to y) \to ((y \to z) \to (x \to z)) = 1,$$

$$x \to y = 1 \;\&\; y \to x = 1 \;\Rightarrow\; x = y.$$

The underlying order: $x \leq y \Leftrightarrow x \to y = 1$.

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Motivation: Stone algebras

Let A be a Stone algebra, i.e., a distributive lattice with pseudocomplementation satisfying

$$x'' \vee x' = 1.$$

Then A is uniquely determined by

$$\langle B(A), D(A), F_A \rangle$$

where

- $B(A) = \{x \in A \mid x'' = x\}$ ... boolean elements,
- $D(A) = \{x \in A \mid x' = 0\}$ ... dense elements,
- the map $F_A \colon B(A) \to \mathrm{Fil}(D(A))$ defined by

$$F_A(a) = \{x \in D(A) \mid x \geq a\}$$

or $F_A(a) = \{x \in D(A) \mid x \geq a'\}$.

Bounded residuated structures

Let A be a bounded residuated structure with least element 0. Let

$$x' = x \to 0.$$

Then

- $D(A) = \{x \in A \mid x' = 0\}$ is a filter, i.e., the kernel $[1]_\theta$ of a relative congruence θ of A;
- $B(A) = \{x \in A \mid x \vee x' = 1\}$ is a boolean subalgebra;
- $R(A) = \{x \in A \mid x'' = x\}$ may be a subalgebra.

Does the triple

$$\langle S(A), D(A), F_A \rangle,$$

where

- $S(A) = B(A)$, or $S(A) = R(A)$ when $R(A)$ is a subalgebra,
- $F_A(a) = \{x \in D(A) \mid x \geq a\}$ for $a \in S(A)$,

characterize A? Sometimes.

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characterize A? Sometimes.

Stonean residuated structures

We call a bounded residuated structure A stonean if it satisfies

$$x'' \vee x' = 1.$$

In this case,

- $B(A) = R(A)$;
- for every $x \in A$,

$$x = x'' \wedge (x'' \to x)$$

  where $x'' \in B(A)$ and $x'' \to x \in D(A)$.

Moreover,

- for every $a \in B(A)$, the map

$$\varrho_a \colon x \in D(A) \mapsto a \to x \in D(A)$$

  is a closure endomorphism of $D(A)$ with

$$\ker(\varrho_a) = \{x \in D(A) \mid a \leq x\} = F_A(a).$$

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Given a stonean BCK-algebra A, let $T = \langle B(A), D(A), E_A \rangle$ where $E_A \colon a \in B(A) \mapsto \varrho_a \in \mathcal{CE}(D(A))$. Let

$$\mathrm{Alg}(T) = \{\langle a, x \rangle \in B(A) \times D(A) \mid x = \varrho_a(x)\}$$

and

$$\langle a, x \rangle \to \langle b, y \rangle = \langle a \to b, x \to \varrho_a(y) \rangle.$$

Then $\langle \mathrm{Alg}(T); \to, \langle 0, 1 \rangle, \langle 1, 1 \rangle \rangle$ is a stonean BCK-algebra with

$$B(\mathrm{Alg}(T)) = \{\langle a, 1 \rangle \mid a \in B(A)\},$$
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Moreover, the map $\eta \colon x \in A \mapsto \langle x'', x'' \to x \rangle \in \mathrm{Alg}(T)$ is an embedding of A into $\mathrm{Alg}(T)$.

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Two triples $T_1 = \langle B_1, D_1, E_1\rangle$ and $T_2 = \langle B_2, D_2, E_2\rangle$ are isomorphic if $g\colon B_1 \cong B_2$, $h\colon D_1 \cong D_2$ and the diagram commutes:

$$\begin{array}{ccc} B_1 & \xrightarrow{\quad g \quad} & B_2 \\ {\scriptstyle E_1}\big\downarrow & & {\scriptstyle E_2}\big\downarrow \\ \mathcal{CE}(D_1) & \xrightarrow{\quad \bar{h} \quad} & \mathcal{CE}(D_2) \end{array}$$

Let A_1, A_2 be stonean BCK-algebras. If the triples $T_1 = \langle B(A_1), D(A_1), E_{A_1}\rangle$ and $T_2 = \langle B(A_2), D(A_2), E_{A_2}\rangle$ are isomorphic, then $\mathrm{Alg}(T_1) \cong \mathrm{Alg}(T_2)$.

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Stonean residuated po-monoids

Given a stonean residuated po-monoid A, let $T = \langle B(A), D(A), E_A \rangle$ where $E_A \colon a \in B(A) \mapsto \varrho_a \in \mathcal{CE}(D(A))$. Let

$$\begin{gathered} \mathrm{Alg}(T) = \{\langle a, x\rangle \in B(A) \times D(A) \mid x = \varrho_a(x)\}, \\ \langle a, x\rangle \to \langle b, y\rangle = \langle a \to b, x \to \varrho_a(y)\rangle, \\ \langle a, x\rangle \cdot \langle b, y\rangle = \langle a \wedge b, \varrho_{a\wedge b}(xy)\rangle. \end{gathered}$$

Then $\big\langle \mathrm{Alg}(T); \cdot, \to, \langle 0,1\rangle, \langle 1,1\rangle \big\rangle$ is a stonean residuated po-monoid with

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Moreover, the map $\eta \colon x \in A \mapsto \langle x'', x'' \to x\rangle \in \mathrm{Alg}(T)$ is an embedding of A into $\mathrm{Alg}(T)$.

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\langle a, x\rangle \wedge \langle b, y\rangle = \langle a \wedge b, \varrho_{a\wedge b}(x \wedge y)\rangle,\\
\langle a, x\rangle \vee \langle b, y\rangle = \langle a \vee b, x\varrho_{a'}(y) \vee y\varrho_{b'}(x)\rangle.
\end{gathered}$$

Then $\big\langle \mathrm{Alg}(T); \vee, \wedge, \cdot, \to, \langle 0, 1\rangle, \langle 1, 1\rangle\big\rangle$ is a stonean residuated ℓ-monoid with

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Divisible residuated structures

A residuated po-monoid is divisible if

$$x(x \to y) = y(y \to x).$$

A BCK-algebra is divisible if it satisfies

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z).$$

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For any divisible bounded BCK-algebra A, $R(A) = \{x \in A \mid x'' = x\}$ is a subalgebra of A and an MV-algebra in its own right. For any $a \in R(A)$, $\varrho_a \colon x \mapsto a \to x$ is a closure endomorphism of $D(A)$. Let $T = \langle R(A), D(A), E_A \rangle$ where $E_A \colon a \in R(A) \mapsto \varrho_a \in \mathcal{CE}(D(A))$, and

$$\mathrm{Alg}(T) = \{\langle a, x \rangle \in R(A) \times D(A) \mid x = \varrho_a(x)\},$$
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Then $\big\langle \mathrm{Alg}(T); \to, \langle 0, 1 \rangle, \langle 1, 1 \rangle \big\rangle$ is a bounded divisible BCK-algebra with

$$R(\mathrm{Alg}(T)) = \{\langle a, 1 \rangle \mid a \in R(A)\},$$
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A BL-algebra is a bounded residuated ℓ-monoid which is divisible and prelinear, i.e. $(x \to y) \vee (y \to x) = 1$. Note:

$$x \wedge y = x(x \to y),$$
$$x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x).$$

Let A be a BL-algebra, $T = \langle R(A), D(A), E_A \rangle$ and

$$\mathrm{Alg}(T) = \{\langle a, x \rangle \in R(A) \times D(A) \mid x = \varrho_a(x)\},$$
$$\langle a, x \rangle \to \langle b, y \rangle = \langle a \to b, x \to \varrho_a(y) \rangle,$$
$$\langle a, x \rangle \cdot \langle b, y \rangle = \langle ab, \varrho_{ab}(xy) \rangle.$$

Then $\langle \mathrm{Alg}(T); \vee, \wedge, \cdot, \to, \langle 0, 1 \rangle, \langle 1, 1 \rangle \rangle$ is a BL-algebra isomorphic to A under $\eta \colon x \in A \mapsto \langle x'', x'' \to x \rangle \in \mathrm{Alg}(T)$.

BL-algebras

A BL-algebra is a bounded residuated ℓ-monoid which is divisible and prelinear, i.e. $(x \to y) \vee (y \to x) = 1$. Note:

$$x \wedge y = x(x \to y),$$
$$x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x).$$

Let A be a BL-algebra, $T = \langle R(A), D(A), E_A \rangle$ and

$$\mathrm{Alg}(T) = \{\langle a, x \rangle \in R(A) \times D(A) \mid x = \varrho_a(x)\},$$
$$\langle a, x \rangle \to \langle b, y \rangle = \langle a \to b, x \to \varrho_a(y) \rangle,$$
$$\langle a, x \rangle \cdot \langle b, y \rangle = \langle ab, \varrho_{ab}(xy) \rangle.$$

Then $\big\langle \mathrm{Alg}(T); \vee, \wedge, \cdot, \to, \langle 0, 1 \rangle, \langle 1, 1 \rangle \big\rangle$ is a BL-algebra isomorphic to A under $\eta \colon x \in A \mapsto \langle x'', x'' \to x \rangle \in \mathrm{Alg}(T)$.

Thank you!