Semigroups of order-preserving transformations of a countable chain

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Stara Lesna 10. September 2014

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Theorem

Any semigroup may be isomorphically embedded into $T(X)$ for a suitable set X.

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- $k(\alpha) := \left|\{x \in im\alpha \mid |x\alpha^{-1}| = \aleph_0\right|$ *infinite contraction index* of α

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Theorem (Gavrilov, 1965)

Let $S \leq T(X)$ with $Sym(X) \leq S$. Then S is maximal if and only if $S = S_i$ for some $i \in \{1, 2, 3, 4, 5\}$.

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- Pinsker (2005) X has arbitarary cardinality

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- the stabiliser of a partition of X into finitely many subsets of equal cardinality

Order-preserving Transformations

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- Here: $X = \mathbb{N}$ and $X = \mathbb{Z}$, respectively.

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Theorem

A semigroup $S \leq IO(\mathbb{N})$ *is maximal if and only if* $S = IO(\mathbb{N}) \setminus \{\alpha\}$ *for some* $\alpha \in IO(\mathbb{N})$ *with* $n\alpha - n \leq 1$ *for all* $n \in \mathbb{N}$.

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Definition

Let $A^{(1)}$ be the set of all $\alpha \in O(\mathbb{N})$ with $1 \in im\alpha$, $1\alpha^{-1} = \{1\}$, and one of the following both properties is satisfied:
1) K_α has maximum ≥ 2 and $\alpha \notin SO(\mathbb{N})$;
2) G_α has no maximum and $\alpha \notin IO(\mathbb{N})$.

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Theorem

Let $S \leq O(\mathbb{N})$ with $A^{(1)} \subseteq S$. Then S is maximal if and only if S has one of the following form

$S = \{\alpha \in O(\mathbb{N}) \mid 2\alpha \geq 2\}$ *or*

$S = \{\alpha \in O(\mathbb{N}) \mid 1\alpha = 1\}$ *or*

$S = (O(\mathbb{N}) \setminus SO(\mathbb{N})) \cup \{id_{\mathbb{N}}\}$ *or*

$S = (O(\mathbb{N}) \setminus IO(\mathbb{N})) \cup \{id_{\mathbb{N}}\}$ *or*

$S = \{\alpha \in O(\mathbb{N}) \mid 2\alpha = 1$ *and* $maxG_\alpha$ *exists* **or** $1\alpha = 1$ *and* $1\alpha^{-1} = \{1\}\}$

or

S *is the set of all* $\alpha \in O(\mathbb{N})$ *such that* $\max K_\alpha$ *does not exist and* $maxG_\alpha$ *exists.*

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Theorem

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