Lattices of annihilators in commutative algebras over fields

Małgorzata Jastrzębska

Siedlce University of Natural Sciences and Humanities

Summer School on General Algebra and Ordered Sets
September 6 – 12, 2014
Stará Lesná

In this notes $\mathbb{K}$ will be a field.

Definition

An algebra over a field $\mathbb{K}$ is a vector space A over $\mathbb{K}$ together with a bilinear associative multiplication. In other words, for arbitrary elements $a, b, c \in A$ and for arbitrary $\lambda \in \mathbb{K}$ the following equalities are satisfied:

1. $a(b + c) = ab + ac$;
2. $(b + c)a = ba + ca$;
3. $(ab)c = a(bc)$;
4. $(\lambda a)b = a(\lambda b) = \lambda(ab)$.

Examples of algebras:

- Any field extension $\mathbb{L} \supseteq \mathbb{K}$,
- For every $1 \leq n < \infty$ the set of all n-by-n matrices over $\mathbb{K}$ with standard operactions,
- The commutative polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ with indeterminates $x_1, \ldots, x_n$ and coefficients in $\mathbb{K}$.
- The noncommutative polynomial ring $\mathbb{K}\{x_1, \ldots, x_n\}$ with indeterminates $x_1, \ldots, x_n$ and coefficients in $\mathbb{K}$.

Agreements

An algebra A is said to be finite dimensional if the space A is finite dimensional over $\mathbb{K}$. All algebras are finite dimensional, with $1 \neq 0$.

If A is an algebra, then by A^{op} we denote the algebra with the same linear structure over $\mathbb{K}$, but with the opposite multiplication.

All lattices considered here have the smallest element ω and the largest element $\Omega \neq \omega$. If P is any partially ordered set (poset), then by P^{op} we denote the set P, but with the reverse order.

A $\mathbb{K}$-subspace I of a $\mathbb{K}$-algebra A is said to be a left ideal if

$$\forall_{a\in A}\forall_{i\in I}\ \ ai \in I \quad (AI \subseteq I).$$

Similarly, a $\mathbb{K}$-subspace $J \subseteq A$ which satisfies

$$\forall_{a\in A}\forall_{j\in J}\ \ ja \in J \quad (JA \subseteq J)$$

is a right ideal in A.

For every algebra A the set $\mathcal{I}_l(A)$ of all left ideals and the set $\mathcal{I}_r(A)$ of all right ideals, ordered by inclusion are complete, modular lattices with operations:

$$I \vee J = I + J \quad \text{and} \quad I \wedge J = I \cap J. \tag{1}$$

A two-sided ideal in an algebra A is a left ideal of A that is also a right ideal of A, and is called an ideal. The set $\mathcal{I}(A)$ of all ideals in A

$$\mathcal{I}(A) = \mathcal{I}_l(A) \cap \mathcal{I}_r(A)$$

ordered by inclusion is a complete, modular lattice.

An algebra A is a local algebra if it has an ideal, which is the unique left maximal ideal in A and the unique right maximal ideal in A. This ideal is known as the radical of A.

Agreements

If $X \subseteq A$ is a subset, then let $\mathrm{L}_A(X) = \mathrm{L}(X)$ be the left annihilator of X in A and let $\mathrm{R}_A(X) = \mathrm{R}(X)$ be the right annihilator of X in A :

$$\mathrm{L}(X) = \{a \in A : aX = 0\}, \tag{2}$$

$$\mathrm{R}(X) = \{a \in A : Xa = 0\}. \tag{3}$$

Let $\mathcal{A}_l(A)$ be the set of all left annihilators in A and $\mathcal{A}_r(A)$ be the set of all right annihilators in A. Then $\mathcal{A}_l(A) \subseteq \mathcal{I}_l(A)$ is a complete lattice with operations:

$$I \vee J = \mathrm{L}(\mathrm{R}(I) \cap \mathrm{R}(J)) \quad \text{and} \quad I \wedge J = I \cap J,$$

for $I, J \in \mathcal{A}_l(A)$, $\omega = 0$ and $\Omega = A$.
Similarly, $\mathcal{A}_r(A) \subseteq \mathcal{I}_r(A)$ is a complete lattice with operations

$$I \vee J = \mathrm{R}(\mathrm{L}(I) \cap \mathrm{L}(J)) \quad \text{and} \quad I \wedge J = I \cap J,$$

for $I, J \in \mathcal{I}_r(A)$, $\omega = 0$ and $\Omega = A$.

Between $\mathcal{A}_l(A)$ and $\mathcal{A}_r(A)$ we have a Galois correspondence:

$$\mathcal{A}_l(A) \stackrel{\mathrm{R}}{\to} (\mathcal{A}_r(A))^{op} \quad \text{and} \quad (\mathcal{A}_r(A))^{op} \stackrel{\mathrm{L}}{\to} \mathcal{A}_l(A).$$

Between $\mathcal{A}_l(A)$ and $\mathcal{A}_r(A)$ we have a Galois correspondence:

$$\mathcal{A}_l(A) \stackrel{\mathrm{R}}{\to} (\mathcal{A}_r(A))^{op} \quad \text{and} \quad (\mathcal{A}_r(A))^{op} \stackrel{\mathrm{L}}{\to} \mathcal{A}_l(A).$$

Theorem

Let A be a finite dimensional algebra over an infinite field $\mathbb{K}$ with $\mathcal{A}_l(A)$ finite. Then A is a finite direct product of local algebras.

Let M be a monoid and let I be an ideal in M. Then the Rees factor monoid M/I is equal to M/ρ, where ρ is the congruence on M given by

$$(s,t) \in \rho \ \text{ if either } \ s = t \ \text{ or } \ s,t \in I. \qquad (4)$$

If M is a monoid, then a monoid algebra $\mathbb{K}[M]$ is a $\mathbb{K}$-space with the basis M and the multiplication induced by the multiplication in M.

If M is a monoid with 0, then $\mathbb{K}0$ is an ideal in $\mathbb{K}[M]$. By contracted monoid algebra of M over $\mathbb{K}$, denoted by $\mathbb{K}_0[M]$, we mean the factor algebra $\mathbb{K}[M]/\mathbb{K}0$.

Example

Let P be a nonempty poset. Then there exists contracted monoid algebra $\mathbb{K}(P)$ such that $P \subset \mathbb{K}(P)$ and $\mathbb{K}(P)$ has a natural gradation given by:

$$\mathbb{K}(P) = \mathbb{K} \oplus V \oplus V^2,$$

where the natural base of V can be identified with P and the natural base of V^2 can be identified with $\{xy : x, y \in P,\ x \not\leq y\}$. If $P = \emptyset$ then $\mathbb{K}(P) = \mathbb{K}$. The algebra $\mathbb{K}(P)$ is a local algebra with the radical $J = V \oplus V^2$ and with the residue field $\mathbb{K}(P)/J = \mathbb{K}$.

If L is a lattice then we put $\mathbb{K}\langle L\rangle = \mathbb{K}(P)$, where $P = L \setminus \{\omega, \Omega\}$.

Theorem

Let P be any finite poset and let $\phi : P \longrightarrow \mathcal{A}_l(\mathbb{K}(P))$ be given by $\phi(x) = \mathrm{L}_{\mathbb{K}(P)}(x)$ for $x \in P$. Then ϕ is an embedding and preserves all existing meets and joins.

If L is a finite lattice, then ϕ extends uniquely to a lattice isomorphism of L with the interval $[\phi(\omega), \phi(\Omega)] \subseteq \mathcal{A}_l(\mathbb{K}\langle L\rangle)$.

Theorem

If P is a poset then every annihilator in $\mathbb{K}(P)$ is an ideal. As a consequence, if L is a lattice, then every annihilator in the algebra $\mathbb{K}\langle L\rangle$ is an ideal, so $\mathcal{A}_l(\mathbb{K}\langle L\rangle) \subseteq \mathcal{I}(\mathbb{K}\langle L\rangle)$ and $\mathcal{A}_r(\mathbb{K}\langle L\rangle) \subseteq \mathcal{I}(\mathbb{K}\langle L\rangle)$.

Let A be a commutative algebra. Then $\mathrm{R}(X) = \mathrm{L}(X)$ for any subset $X \subseteq A$. Thus $\mathcal{A}_l(A) = \mathcal{A}_r(A)$. We put $\mathrm{R}(X) = \mathrm{L}(X) = \mathrm{a}(X)$ and $\mathcal{A}_l(A) = \mathcal{A}_r(A) = \mathcal{A}(A)$. Then $X \longrightarrow \mathrm{a}(X)$ is an antiautomorphism of the lattice $\mathcal{A}(A)$.

If a lattice has exactly one atom (coatom) then we will denote it by $\hat{\omega}$ (respectively $\hat{\Omega}$).

Basic Example in Commutative case

Let P be a nonempty poset and let P' be the set such that $|P| = |P'|$. Let $f : P \longrightarrow P'$ be a bijection given by $f(x) = x' \in P'$. Put $P^* = P \cup P'$. Let $S(P)$ be the free, commutative monoid with the set P^* of free generators.

Consider in $S(P)$ an ideal I generated by all products xyz where $x, y, z \in P^*$ and by all elements of the set $\{x'y \mid x, y \in P \text{ where } x \leq y\}$. Put $\overline{S} = S(P)/I$, the Rees factor monoid.

Clearly $P^* \subseteq \overline{S}$ in a natural way and $(P^*)^2 = \{0\} \cup \{xy | x, y \in P\} \cup \{x'y' | x', y' \in P'\} \cup \{x'y \mid x, y \in P, x \not\leq y\}$. Moreover, $\overline{S} = \{1\} \cup P^* \cup (P^*)^2$.

Now let $\mathbb{K}((P)) = \mathbb{K}_0[\overline{S}]$ be the contracted monoid algebra.
Thus $P \subset \mathbb{K}((P))$ and $\mathbb{K}((P))$ has the natural gradation given by:

$$\mathbb{K}((P)) = \mathbb{K} \oplus W \oplus W^2, \tag{5}$$

where the natural base of W can be identified with P^* and the natural base of W^2 can be identified with $(P^*)^2 \setminus \{0\}$.
Our algebra $\mathbb{K}((P))$ is a local, commutative algebra with the radical $J = W \oplus W^2$ and with the residue field $\mathbb{K}((P))/J = \mathbb{K}$.

Commutative case

If L is a lattice, then let us take $\mathbb{K}\langle\langle L\rangle\rangle = \mathbb{K}((P))$, where $P = L \setminus \{\Omega, \hat{\Omega}, \omega, \hat{\omega}\}$ if $\hat{\Omega}, \hat{\omega}$ exist.
Using the above notations we have

Theorem

Let P be nonempty finite poset and let $\psi : P \longrightarrow \mathcal{A}(\mathbb{K}((P)))$ be given by $\psi(x) = \mathrm{a}_{\mathbb{K}((P))}(x)$ for $x \in P$. Then ψ is an embedding and preserves all existing meets and joins.
If L is a finite lattice and L is none of chains with at most 4 elements then ψ extends uniquely to a lattice embedding of L into $\mathcal{A}(\mathbb{K}\langle\langle L\rangle\rangle)$.

Corollary

Every finite lattice can be represented as a sublattice of lattice of annihilators in a commutative $\mathbb{K}$-algebra.

Thank you for your attention!

- Małgorzata Jastrzębska, Jan Krempa, *On lattices of annihilators*, (to appear).
- Dorota Niewieczerzał, *Some examples of rings with annihilator conditions*, Bull. Acad. Polon. Sci. Math. 26 (1978), no. 1, 1–5. MR0485978 (58 #5770).
- Roman R. Zapatrin, *Representation of finite lattices by annihilators of completely 0-simple semigroups*, Semigroup Forum 59 (1999), no. 1, 121–125. MR1847947.