Algebraic Structures in Quantum Structures

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- $\delta_\omega(A) = 1$ iff $\omega \in A$ otherwise $= 0$
- the set probability measures $\mathcal{P}(\mathcal{S}) \neq \emptyset$
- observable: $f : \Omega \to \mathbb{R}$, s.t. $f^{-1}(E) \in \mathcal{S}$, $E \in \mathcal{B}(\mathbb{R})$ - measurable

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- $$f(\omega) = \begin{cases} \inf\{r_j : \omega \in x_{r_j}\} & \text{if } \omega \in \bigcup_n A_n, \\ 0 & \text{if } \omega \notin \bigcup_n A_n. \end{cases}$$

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- G. Birkhoff and J. von Neumann, 1936 quantum logic

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Boolean Algebras

A system $A = (A; \vee, \wedge, ', 0, 1)$ is a **Boolean algebra** if type $(2, 2, 1, 0, 0)$ if for all $a, b, c \in A$ we have

1. $a \vee b = b \vee a$, $a \wedge b = b \wedge a$ (commutativity)
2. $(a \vee b) \vee c = a \vee (b \vee c)$, $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (associativity)
3. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c),$
   $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (distributivity)
4. $a \vee a' = 1$, $a \wedge a' = 0$
5. $a \wedge 1 = a = a \vee 0$

Examples

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- Let Ω-topological space, $\mathcal{A}$- the set of all clopen subsets.

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  S(A) and Ext(A) are compact Hausdorff
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- a topological space Ω is **totally disconnected** if
  there exists a base consisting of clopen sets.

Theorem 0.1 (Stone Theorem) *Every Boolean algebra* $A = (A; \vee, \wedge, ', 0, 1)$ *is isomorphic to the Boolean algebra of clopen subsets of a compact, totally disconnected Hausdorff topological space (= Stone space).*

Boolean σ-algebras

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Theorem 0.3 (Loomis-Sikorski) *Every Boolean σ-algebra is a σ-homomorphic image of a σ-algebra of sets.*

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- X is said to be *basically disconnected* provided the closure of every open F_σ subset of X is open.

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(v) $b = a \vee (b \wedge a^{\perp})$ whenever $a \leq b$ (orthomodular law).

- H- Hilbert space,

  $L(H) = \{M \subseteq H : M-$closed subspace of $H\}$

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- given a system of mutually compatible elements, there is a maximal system of mutually compatible elements of L - it is a Boolean algebra

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- $com(x, y) = \bigwedge\{com(a_1, \ldots, a_n) : a_1, \ldots, a_n \in \mathcal{R}(x) \cup \mathcal{R}(y), n \geq 1\}$ =0, = 1, strictly between 0 and 1

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- A, B hermitian operators are compatible iff
  $AB = BA$

States and Greechie Diagrams

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- finite sequence $\{B_0, \ldots, B_{n-1}\}$ from $\mathcal{B}$ is a loop of order n $(n \geq 3)$ if
- (i) $\forall\ i \in \{0, 1, \ldots, n-1\}$ we have $B_i \cap B_{i+1} = \{0, 1, x, x^\perp\}$ x atom in both BAs

- (ii) if $j \notin \{i-1, i, i+1\}$, $B_i \cap B_j = \{0, 1\}$

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- **Theorem 0.8** *If $\mathcal{B}$ is a system of almost disjoint system of BAs, then $L = \bigcup\{B : B \in \mathcal{B}\}$ is (1) an OMP iff $\mathcal{B}$ doesn't contain any loop of order 3*
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- There is a finite stateless OMP

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- or iff $a + b$, $b + c$ and $a + c$ exist, then $a + b + c$ is defined in A

Firefly Examples of quantum structures

Fig. 4.1

Firefly Examples of quantum structures

Fig. 4.1

- The experiment A: Look at the front window.
  The experiment B: Look at the side window.
  The outcomes of A and B are:

- See a light in the left half (l_A, l_B), right half (r_A, r_B) of the window or see no light (n_A, n_B). It is clear that $n_A = n_B =: n$ and we put $l_A =: l$, $r_A =: r$, $l_B =: f$, $r_B =: b$ (f for the front, b for the back)

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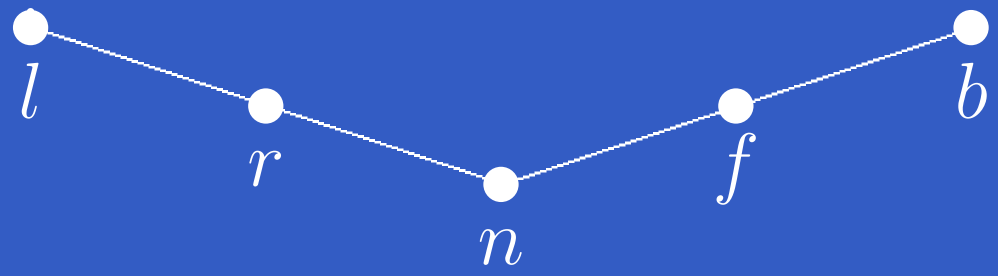
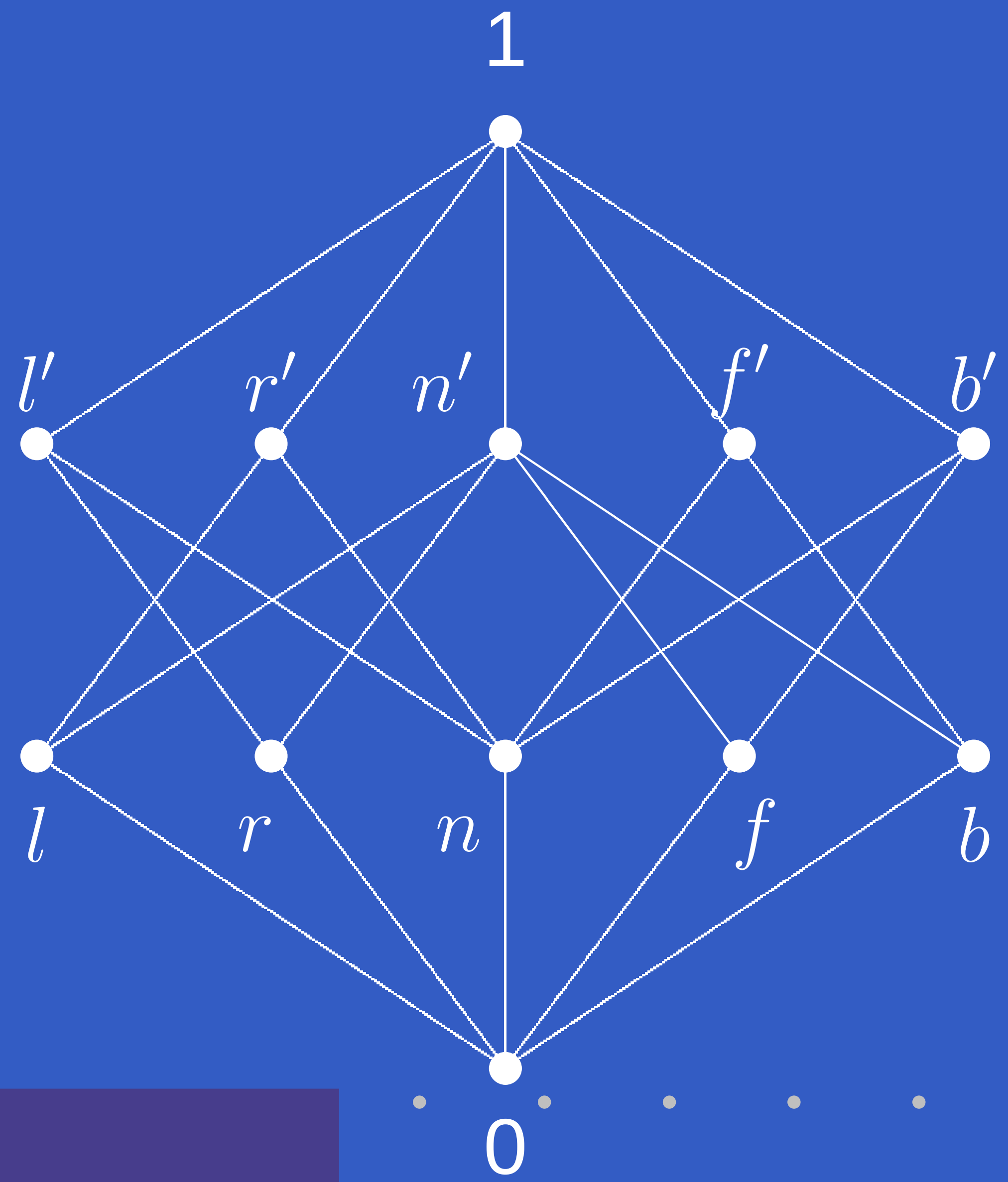


Fig. 4.2

-


1
l'
r'
n'
f'
b'
l
r
n
f
b
0

Three-chamber box

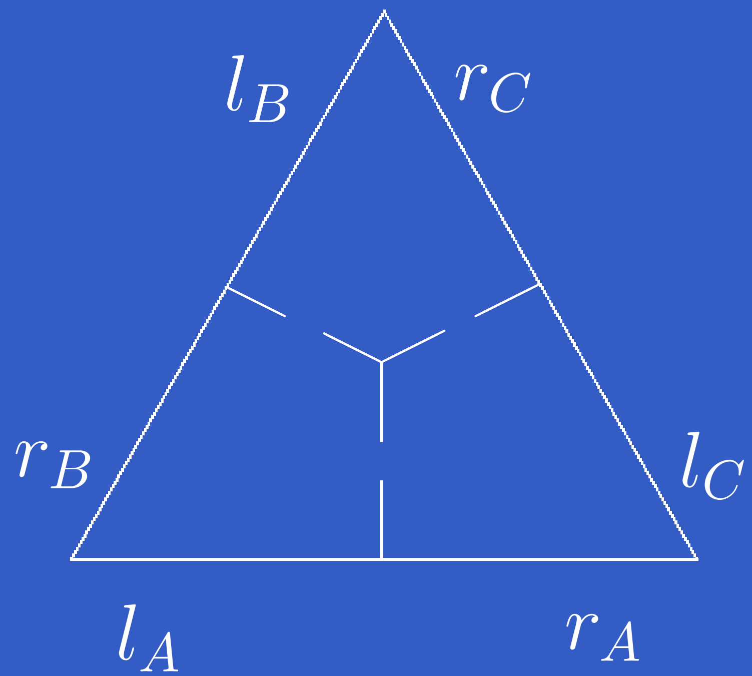


Fig. 4.5

Three-chamber box

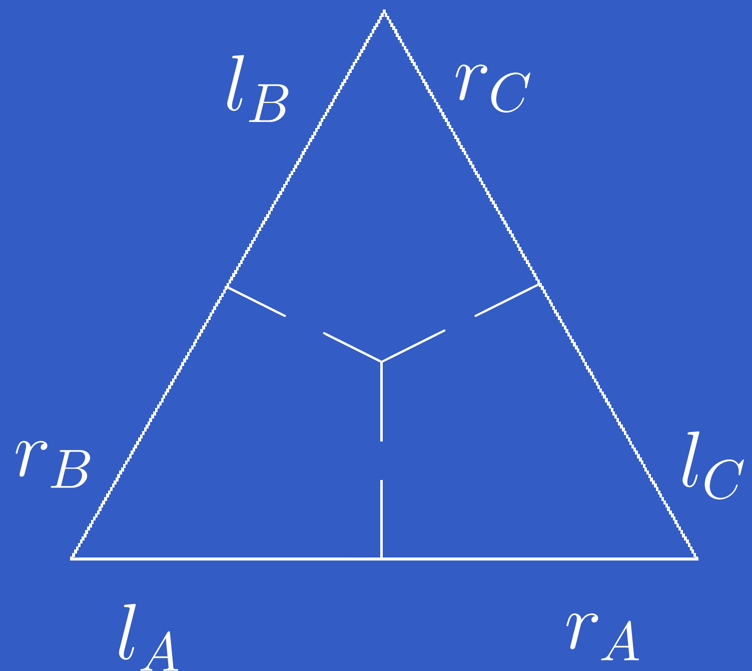


Fig. 4.5

- three experiments, corresponding to the three windows A, B and C. we record l_E, r_E, n_E if we see, respectively, a light to the left, right, of the center line or no light.

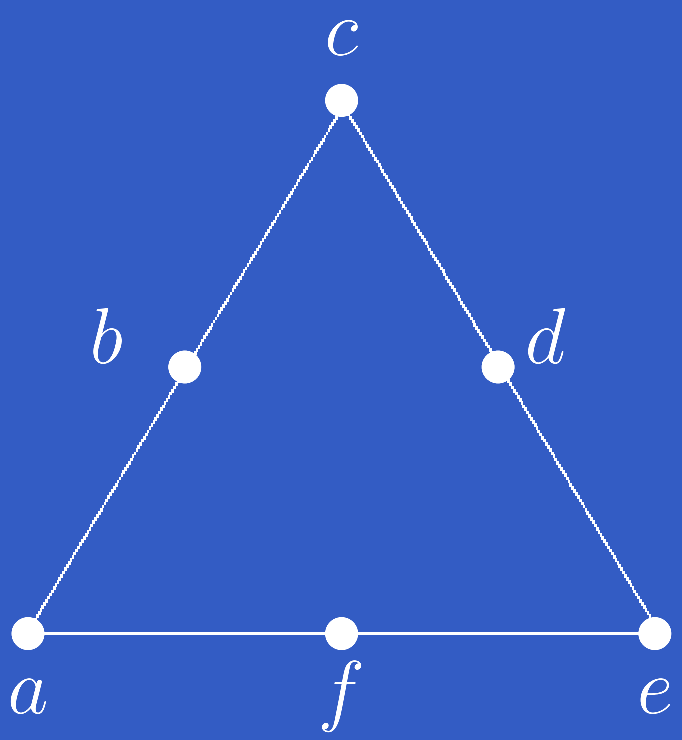
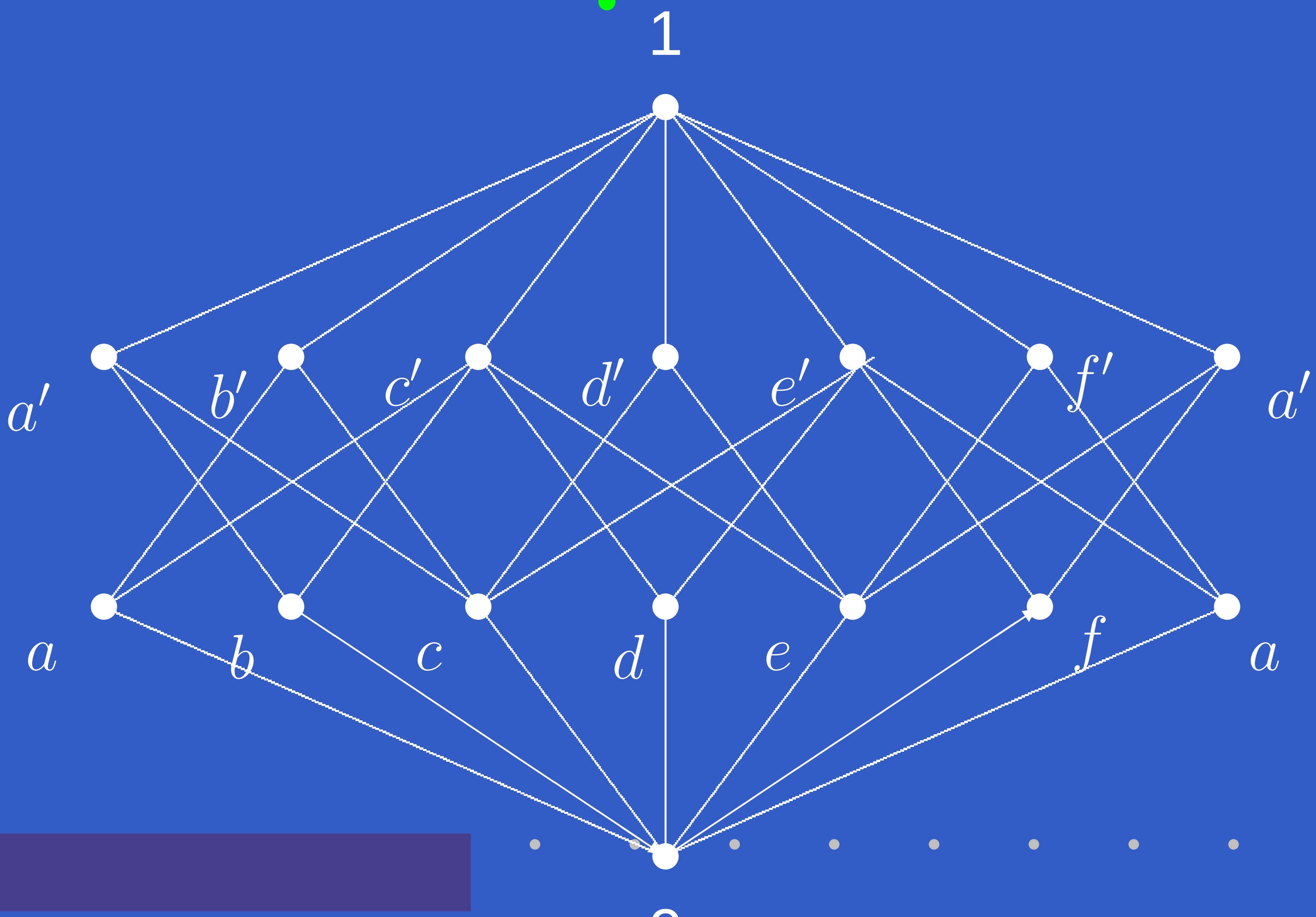


Fig. 4.6 Wright triangle


1
a′
b′
c′
d′
e′
f′
a′
a
b
c
d
e
f
a

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Examples

$[0,1]$ + restricted from $[0,1]$

po-group $(G; \leq, +, -, 0)$

$$a \leq b \quad \rightarrow \quad a + c \leq b + c$$

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interval EA: $E := \Gamma(G, u)$

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$s(1) = 1.$

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- G - interpolation group whenever $a_1, a_2 \leq b_1, b_2 \; \exists \; c \in G$ s.t. $a_1, a_2 \leq c \leq b_1, b_2$

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- every interval EA has a state

Many-valued Reasoning

- Ulam-game, Pinocchio, Game with black-and white marbles, error correcting codes.
- set, fuzzy set $f: \Omega \to [0,1]$, $f: \Omega \to \{0,1\}$.

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MV-algebra is an algebra $M = (M; \oplus, \odot, ^*, 0, 1)$ of type (2,2,1,0,0) such that, for all $a, b, c \in M$, we have

(i) $a \oplus b = b \oplus a$;
(ii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
(iii) $a \oplus 0 = a$;
(iv) $a \oplus 1 = 1$;
(v) $(a^*)^* = a$;
(vi) $a \oplus a^* = 1$;
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1. $a \vee b = (a^* \oplus b)^* \oplus b$. M is a distributive lattice

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- **Bold algebra** $\mathcal{F} \subseteq [0,1]^\Omega$ **(i)** $1 \in \mathcal{F}$, **(ii)** $f \in \mathcal{F}$, then $1 - f \in \mathcal{F}$, **(iii)** $f, g \in \mathcal{F}$, and

$$(f \oplus g)(\omega) := \min\{f(\omega) + g(\omega), 1\}, \omega \in \Omega,$$

then $f \oplus g \in \mathcal{F}$.

$$(f \odot g)(\omega) := \max\{0, (f(\omega) + g(\omega) - 1)\}$$

- Let $(G, +, 0, \leq)$ be an ℓ-group, i.e. a group such that if $a \leq b$, $a, b \in G$, then for any $c \in G$, $c + a \leq c + b$, and G is a lattice.

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- (G, u) ℓ-group with strong unit.
- $\Gamma(G, u) = [0, u]$

$$a \oplus b = (a + b) \wedge u, a, b \in \Gamma(G, u),$$

$$a \odot b = 0 \vee (a + b - u), a, b \in \Gamma(G, u)$$

- $(\Gamma(G,u);\oplus,\odot,^*,0,u)$ is an MV-algebra, where $a^* = u - a$.

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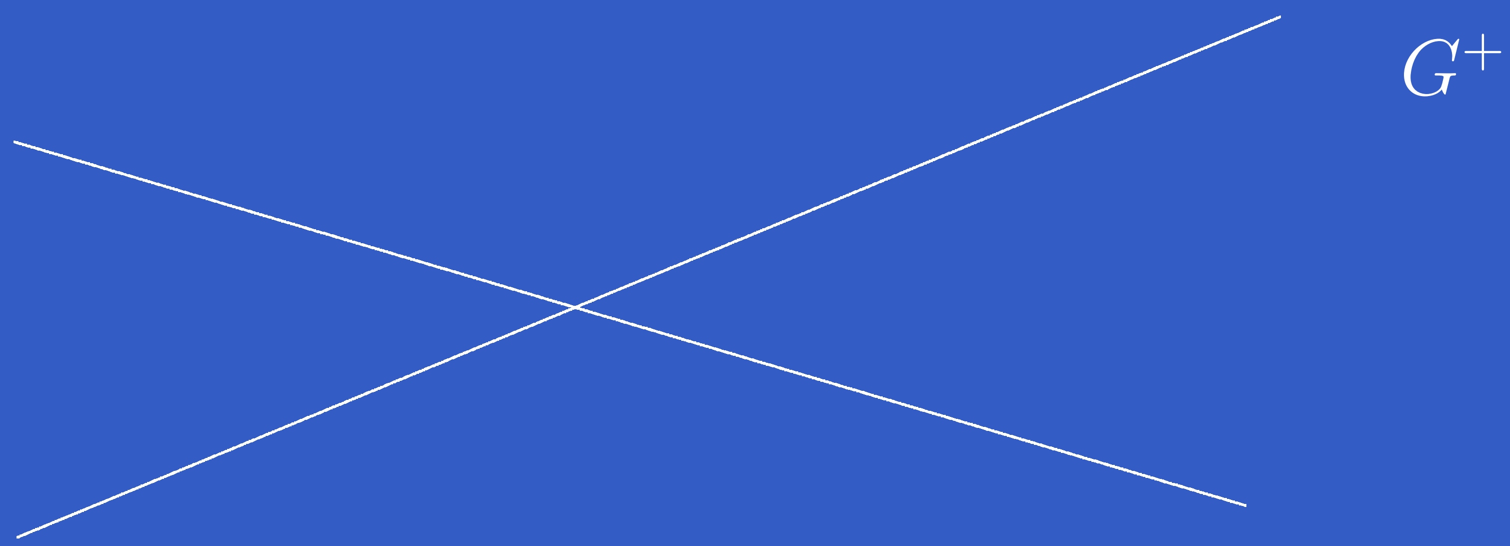

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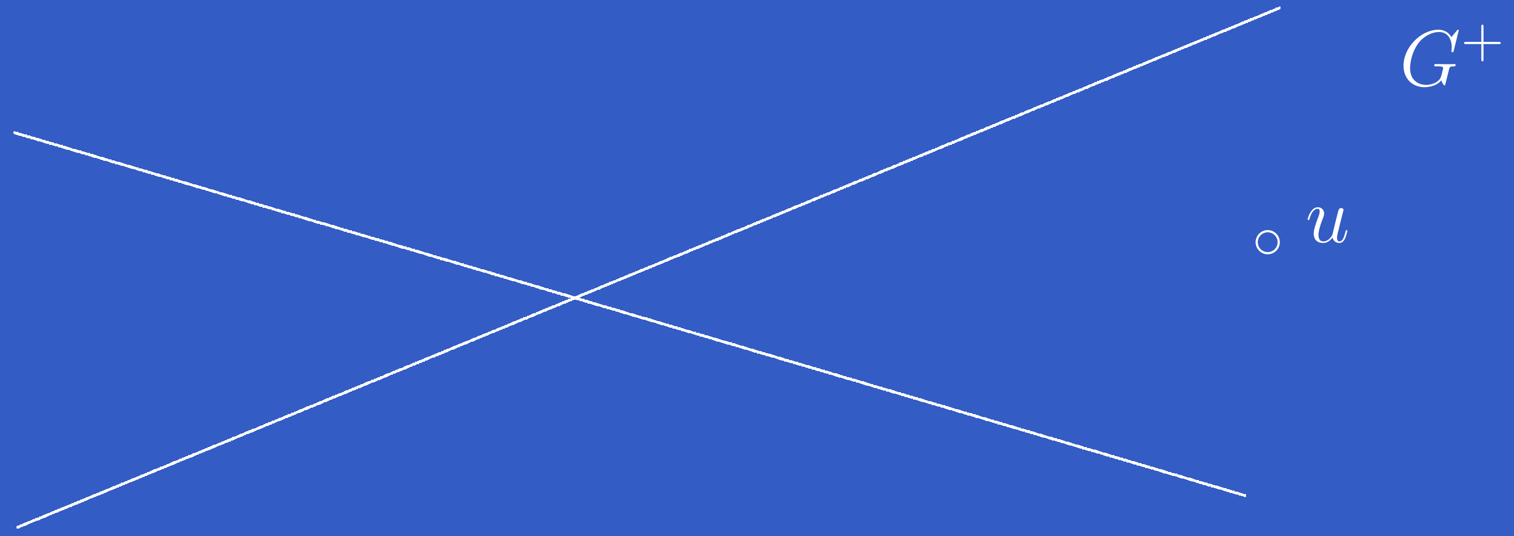

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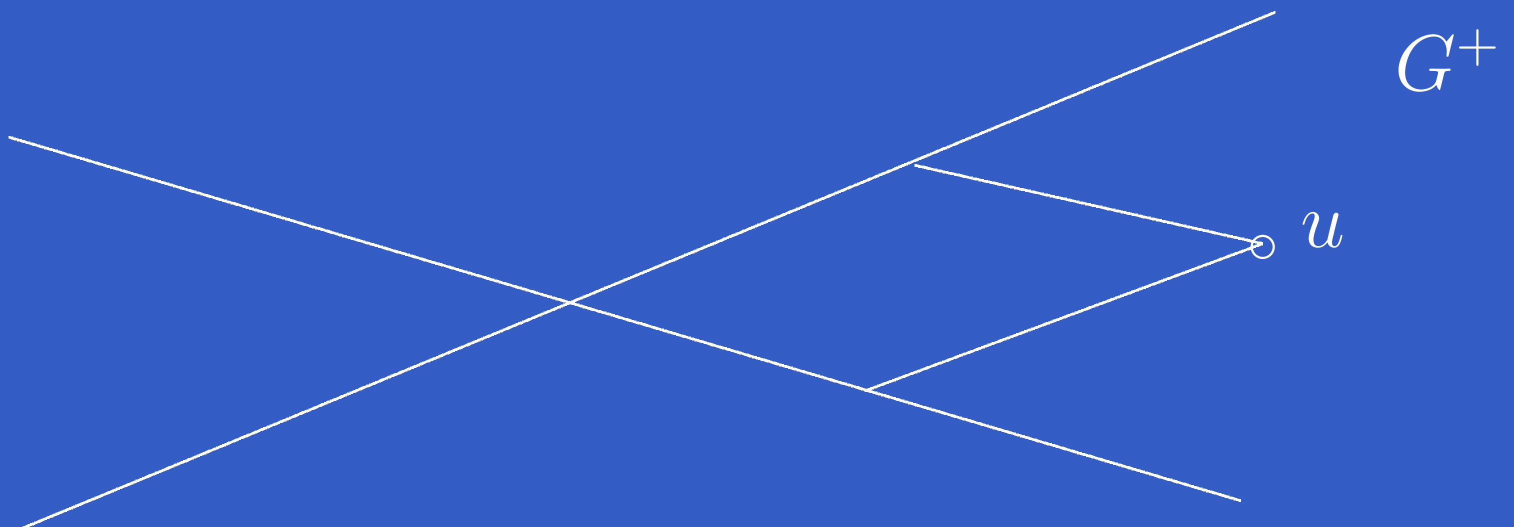

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- Every lattice ordered EA can be covered by sub MV-effect algebras, not true for every EA

States on MV-algebras

- **state** $s : M \to [0,1]$ such that (i) $s(1) = 1$ and (ii) $s(a+b) = s(a) + s(b)$ if $a \leq b^*$.

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- $\mathcal{S}(E) \cong \mathcal{S}(Aff(\mathcal{S}(E)), 1)$, $s \mapsto f(s)$, $f \in$ Aff$(\mathcal{S}(E))$

Simplices vs EAs

- convex cone- in a real linear space V is any subset C of V such that (i) $0 \in C$, (ii) if $x_1, x_2 \in C$, then $\alpha_1 x_1 + \alpha_2 x_2 \in C$ for any $\alpha_1, \alpha_2 \in \mathbb{R}^+$.

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- strict cone- is any convex cone C such that $C \cap -C = \{0\}$,
- base- for a convex cone C is any convex subset K of C $y \in C \setminus \{0\}$ may be uniquely expressed in the form $y = \alpha x$ for some $\alpha \in \mathbb{R}^+$, $x \in K$

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- $\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & 1-\beta_1 \end{pmatrix}$, the parameters β_1 and β_2 must satisfy the inequality $(\beta_1 - \frac{1}{2})^2 + \beta_2^2 \leq \frac{1}{4}$, and vice-versa. Hence, the state space is affinely isomorphic with the latter circle. The state space for $H = \mathbb{C}^2$ is affinely homeomorphic with a three-dimensional real sphere

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- $\dim H = 2$, regular states $\cong$ unit ball in $\mathbb{R}^2$

Structure of the state space

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- A convex compact Hausdorff space $K \neq \emptyset$ is affinely isomorphic to the state space of some MV-algebra iff K is a Bauer simplex.

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- A convex compact Hausdorff space $K \neq \emptyset$ is affinely isomorphic to the state space of some EA with (RDP) iff K is a Choquet simplex
- there is no MV-algebra whose state space is affinely isomorphic to the closed square or to the closed unit circle

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States vs Integrals

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- **Theorem 0.10** *Let E be an effect algebra with RDP and let s be a state on E. Then there is a unique maximal regular Borel probability measure $\mu_s \sim \delta_s$ on $\mathcal{B}(\mathcal{S}(E))$ such that*

$$s(a) = \int_{\mathcal{S}(E)} \hat{a}(x)\, \mathrm{d}\mu_s(x), \quad a \in E.$$

- **Theorem 0.11** *Let* $E = \Gamma(G, u)$ *be an interval effect algebra where* (G, u) *is a unigroup, and let* $\mathcal{S}(E)$ *be a simplex. If* s *is* σ*-additive, then its unique extension,* $\hat{s}$*, on* (G, u) *is* σ*-additive.*

- **Theorem 0.12** *Let E be an MV-algebra and let s be a state on E. Then there is a unique regular Borel probability measure, μ_s, on $\mathcal{B}(\mathcal{S}(E))$ such that $\mu_s(\partial_e \mathcal{S}(E)) = 1$ and*

$$s(a) = \int_{\partial_e \mathcal{S}(E)} \hat{a}(x) \, \mathrm{d}\mu_s(x), \quad a \in E.$$

- **Corollary 0.13** *Let s be a state on an effect algebra E. There is a regular Borel probability measure, μ_s, on the Borel σ-algebra $\mathcal{B}(\mathcal{S}(E))$ such that*

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- **Corollary 0.14** *Let s be a state on an effect algebra E. There is a regular Borel probability measure, μ_s, on the Borel σ-algebra $\mathcal{B}(\mathcal{S}(E))$ such that*

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- **Corollary 0.15** *Let s be a state on an effect algebra E. There is a regular Borel probability measure, μ_s, on the Borel σ-algebra $\mathcal{B}(\mathcal{S}(E))$ such that*

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- $\mathcal{E}(H)$ is isomorphic to an effect-tribe: $\mathcal{E}(H)$ no RDP
- $\Omega(H) = \{\phi \in H : ||\phi|| = 1\}$, $A \in \mathcal{E}(H)$, $\mu_A(\phi) := (A\phi, \phi)$, $\phi \in \Omega(H)$. $\mathcal{T}(H) = \{\mu_A : A \in \mathcal{E}(H)\}$

Loomis-Sikorski theorems

- **Theorem 0.16** *Every σ-MV-algebra is a σ-homomorphic image of a tribe of fuzzy sets.*

Loomis-Sikorski theorems

- **Theorem 0.18** *Every σ-MV-algebra is a σ-homomorphic image of a tribe of fuzzy sets.*
- **Theorem 0.19** *For every monotone σ-complete effect algebra E with RDP, there are a nonempty set Ω, an effect-tribe $\mathcal{T} \subseteq [0,1]^{\Omega}$ with* RDP, *and a σ-homomorphism h from $\mathcal{T}$ onto E.*

New Trends

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GMV-algebras

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GMV-algebras

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- PMV-algebra or GMV-algebra is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z;$

(A2) $x \oplus 0 = 0 \oplus x = x;$

(A3) $x \oplus 1 = 1 \oplus x = 1;$

(A4) $1^{\sim} = 0; 1^{-} = 0;$

(A5) $(x^{-} \oplus y^{-})^{\sim} = (x^{\sim} \oplus y^{\sim})^{-};$

(A6) $x \oplus (x^{\sim} \odot y) = y \oplus (y^{\sim} \odot x) = (x \odot y^{-}) \oplus y = (y \odot x^{-}) \oplus x;$

(A7) $x \odot (x^{-} \oplus y) = (x \oplus y^{\sim}) \odot y;$

(A8) $(x^{-})^{\sim} = x.$

- $$x \leq y \quad \text{iff} \quad x^{-} \oplus y = 1$$

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- $x \vee y = x \oplus (x^\sim \odot y)$ and $x \wedge y = x \odot (x^- \oplus y)$.
- GMV-algebra M is an MV-algebra iff $x \oplus y = y \oplus x$ for all $x, y \in M$.

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$$\begin{aligned} x \oplus y &:= (x + y) \wedge u, \\ x^{-} &:= u - x, \\ x^{\sim} &:= -x + u, \\ x \odot y &:= (x - u + y) \vee 0, \end{aligned}$$

(G,u) unital ℓ-group, u strong unit

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$(\Gamma(G,u); \oplus, ^{-}, ^{\sim}, 0, u)$ is a GMV-algebra.

- **Theorem 0.20** [Dvu 2002] *For any GMV-algebra* M*, there exists a unique* $($*up to isomorphism*$)$ *unital* ℓ*-group* G *with a strong unit* u *such that* $M \cong \Gamma(G, u)$*.*
The functor Γ *defines a categorical equivalence between the category of GMV-algebras and the category of unital* ℓ*-groups.*

- **Theorem 0.21** [Dvu 2002] *For any GMV-algebra M, there exists a unique $($up to isomorphism$)$ unital ℓ-group G with a strong unit u such that $M \cong \Gamma(G, u)$.*
  The functor Γ defines a categorical equivalence between the category of GMV-algebras and the category of unital ℓ-groups.
- $\Gamma(\mathbb{Z} \overrightarrow{\times} G, (1, 0))$ - GMV-algebra such that $x^{\sim} = x^{-}$ (symmetric) but not necessarily MV-algebra

- Let u be the translation $u(t) = t + 1$, $t \in \mathbb{R}$,

$\mathrm{BAut}(\mathbb{R}) = \{g \in \text{Aut}(\mathbb{R}) : \exists\, n \in \mathbb{N},\ \ u^{-n} \leq g \leq u^n\}$.

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- If $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$.
- If $a + b$ and $a + c$ exist and are equal, then $b = c$. If $b + a$ and $c + a$ exist and are equal, then $b = c$.

Pseudo Effect Algebras

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- $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist, and in this case, $(a+b)+c=a+(b+c)$.
- If $a+b$ exists, there are elements $d,e\in E$ such that $a+b=d+a=b+e$.
- If $a+b$ and $a+c$ exist and are equal, then $b=c$. If $b+a$ and $c+a$ exist and are equal, then $b=c$.
- If $a+b$ exists and $a+b=0$, then $a=b=0$.

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- PEA is an EA iff $+$ is commutative
- RDP: $a_1 + a_2 = b_1 + b_2$, there are four elements $c_{11}, c_{12}, c_{21}, c_{22}$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$, and $b_2 = c_{21} + c_{22}$.

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- (RDP)$_1$: RDP + $x \leqslant c_{12}$ and $y \leqslant c_{21}$, we have $x+y$, $y+x$ exists in E and $x+y=y+x$,

- RDP$_2$: RDP + $d_2 \wedge d_3 = 0$ - pseudo MV-algebra

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- AD+Vetterlein: The category of pseudo effect algebras with RDP$_1$ is categorically equivalent with the category of unital po-group with RDP$_1$

States on PEAs

- **Theorem 0.22** *If E is a pseudo effect algebra with* (RDP)*, then either $\mathcal{S}(E)$ is empty or it is a nonempty Choquet simplex.*
  If, in addition, E satisfies (RDP)$_2$*, then either $\mathcal{S}(E)$ is empty or it is a nonempty Bauer simplex.*

States on PEAs

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States on PEAs

- **Theorem 0.24** *If E is a pseudo effect algebra with* (RDP), *then either $\mathcal{S}(E)$ is empty or it is a nonempty Choquet simplex.*
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- Extremal states for GMV-algebras similar as those for MV-algebras
- Representation of states by integral as those for states on EAs

Pseudo BL-algebras

- pseudo BL-algebra - an algebra
  $M = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$

Pseudo BL-algebras

- pseudo BL-algebra - an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$
- (i) $(M; \odot, 1)$ is a monoid (not neces. comm.), $\odot$ is associative with neutral element 1.

  (ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice.

  (iii) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$ $x, y \in M$.

  (iv) $(x \rightarrow y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x),\ x, y \in M.$

  (v) $(x \rightarrow y) \vee (y \rightarrow x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow y),\ x, y \in M.$

ℓ-groups

- if $x \odot y = y \odot x$, M- BL-algebra (Hajek)

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- Is every pseudo BL-algebra good ?
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- $(G^{+})^{J} \uplus (G^{-})^{I}$,
- $x \leq y$ for all $x \in (G^{+})^{J}$, $y \in (G^{-})^{I}$,

Kites

- $a_i^{-1}, b_i^{-1}, \ldots$ for co-ordinates of elements of $(G^{-})^{I}$

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- $1 = (e^{-1})^I := \langle e^{-1} : i \in I \rangle$ $0 = e^J = \langle e : j \in J \rangle$
- $\langle a_i^{-1} : i \in I \rangle \rightsquigarrow \langle b_i^{-1} : i \in I \rangle = \langle a_i b_i^{-1} \wedge e^{-1} : i \in I \rangle$

Kites

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- $\langle a_i^{-1} : i \in I \rangle \to \langle b_i^{-1} : i \in I \rangle = \langle b_i^{-1} a_i \wedge e^{-1} : i \in I \rangle$

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$$\langle f_j : j \in J \rangle \odot \langle a_i^{-1} : i \in I \rangle = \langle f_j a_{\rho(j)}^{-1} \vee e : j \in J \rangle$$

$$\langle f_j : j \in J \rangle \odot \langle g_j : j \in J \rangle = \langle e : j \in J \rangle = 0.$$

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- Theorem: $K_{I,J}^{\lambda,\rho}(\mathbf{G})$ - pseudo BL-algebra

Properties of Kites

- **Lemma 0.25** $K_{I,J}^{\lambda,\rho}(\mathrm{G})$ *is a pseudo MV-algebra if and only if* $\lambda(J) = I = \rho(J)$*.*

Properties of Kites

- **Lemma 0.27** $K_{I,J}^{\lambda,\rho}(\mathrm{G})$ *is a pseudo MV-algebra if and only if* $\lambda(J) = I = \rho(J)$.
- **Lemma 0.28** $K_{I,J}^{\lambda,\rho}(\mathrm{G})$ *is a good pseudo BL-algebra if and only if* $\lambda(J) = \rho(J)$.

Properties of Kites

- **Lemma 0.29** $K_{I,J}^{\lambda,\rho}(\mathrm{G})$ *is a pseudo MV-algebra if and only if* $\lambda(J) = I = \rho(J)$*.*
- **Lemma 0.30** $K_{I,J}^{\lambda,\rho}(\mathrm{G})$ *is a good pseudo BL-algebra if and only if* $\lambda(J) = \rho(J)$*.*
- There are pseudo BL-algebras that are not good

Properties of Kites

- **Lemma 0.31** $K^{\lambda,\rho}_{I,J}(\mathrm{G})$ *is a pseudo MV-algebra if and only if* $\lambda(J) = I = \rho(J)$.
- **Lemma 0.32** $K^{\lambda,\rho}_{I,J}(\mathrm{G})$ *is a good pseudo BL-algebra if and only if* $\lambda(J) = \rho(J)$.
- There are pseudo BL-algebras that are not good
- If G is an Abelian ℓ-group, $K^{\lambda,\rho}_{I,J}(\mathrm{G})$ can be noncommutative

Examples of kites

- $K^{\emptyset,\emptyset}_{0,0}(\mathbf{G})$ is the two-element Boolean algebra for any ℓ-group $\mathbf{G}$

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- $\mathbf{O}$ be the trivial ℓ-group. Then $\lambda = id = \rho$, and $K^{id,id}_{I,J}(\mathbf{O})$ is the two-element Boolean algebra for any choice of I and J.

Examples of kites

- $K_{0,0}^{\emptyset,\emptyset}(\mathbf{G})$ is the two-element Boolean algebra for any ℓ-group $\mathbf{G}$
- $\mathbf{O}$ be the trivial ℓ-group. Then $\lambda = id = \rho$, and $K_{I,J}^{id,id}(\mathbf{O})$ is the two-element Boolean algebra for any choice of I and J.
- if $I = 2$, $J = 1$, $\lambda(0) = 0$, and $\rho(0) = 1$, we get an algebra $K_{2,1}^{\lambda,\rho}(\mathbf{G})$, Jipsen-Montagna

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- $K_{2,1}^{\lambda,\rho}(\mathbb{Z})$ is not good

Examples of kites

- $K_{0,0}^{\emptyset,\emptyset}(\mathbf{G})$ is the two-element Boolean algebra for any ℓ-group $\mathbf{G}$
- $\mathbf{O}$ be the trivial ℓ-group. Then $\lambda = id = \rho$, and $K_{I,J}^{id,id}(\mathbf{O})$ is the two-element Boolean algebra for any choice of I and J.
- if $I = 2$, $J = 1$, $\lambda(0) = 0$, and $\rho(0) = 1$, we get an algebra $K_{2,1}^{\lambda,\rho}(\mathbf{G})$, Jipsen-Montagna
- $K_{2,1}^{\lambda,\rho}(\mathbb{Z})$ is not good
- $K_{n+1,n}^{\lambda,\rho}(\mathbb{Z})$ with $\lambda(i) = i$ and $\rho(i) = i + 1$, for an arbitrary n

Subdirectly irreducible kites

- **Theorem 0.33** *Let* $\mathbf{G}$ *be an* ℓ*-group, and* $K^{\lambda,\rho}_{I,J}(\mathbf{G})$ *a kite. The following are equivalent:*

1. $\mathbf{G}$ *is subdirectly irreducible and for all* $i, j \in I$ *there exists* $m \in \omega$ *such that* $(\rho \circ \lambda^{-1})^m(i) = j$ *or* $(\lambda \circ \rho^{-1})^m(i) = j$*.*
2. $K^{\lambda,\rho}_{I,J}(\mathbf{G})$ *is subdirectly irreducible.*

Subdirectly irreducible kites

- **Theorem 0.35** *Let* $\mathbf{G}$ *be an* ℓ*-group, and* $K^{\lambda,\rho}_{I,J}(\mathbf{G})$ *a kite. The following are equivalent:*

  1. $\mathbf{G}$ *is subdirectly irreducible and for all* $i, j \in I$ *there exists* $m \in \omega$ *such that* $(\rho \circ \lambda^{-1})^m(i) = j$ *or* $(\lambda \circ \rho^{-1})^m(i) = j$*.*
  2. $K^{\lambda,\rho}_{I,J}(\mathbf{G})$ *is subdirectly irreducible.*

- **Lemma 0.36** *Let* $K^{\lambda,\rho}_{I,J}(\mathbf{G})$ *be a subdirectly irreducible kite. Then,* I *and* J *are at most countably infinite.*

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