Closure operators on posets and lattice-valued up-sets

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In applications, we use the results from [2] (Horvath, Šešelja and Tepavčević). In this paper, a lattice induced threshold function was introduced, as a Boolean function determined by a special linear combination of lattice elements. Using these linear combinations, properties of lattice induced threshold functions were investigated, and some particular lattice-valued functions on finite Boolean lattices were represented.

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We prove that a lattice valued up-set, which is a special isotone function from a finite distributive lattice into a complete lattice, is representable by a lattice linear combination if and only if this function is a zero preserving join homomorphism.

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As an application, we deal with a particular representation of isotone functions from a finite distributive lattice into a complete lattice. These functions are called lattice valued up-sets, and the representation is formulated in terms of lattice polynomials.

We prove that a lattice valued up-set, which is a special isotone function from a finite distributive lattice into a complete lattice, is representable by a lattice linear combination if and only if this function is a zero preserving join homomorphism.

We also give conditions for a representation of a lattice valued up-set by a lattice linear combination in case of an arbitrary (not necessarily distributive) finite lattice.

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A mapping $X \mapsto \overline{X}$ on a power set $\mathcal{P}(A)$ of a set A, is a **closure operator** on A, if it fulfills: $X \subseteq \overline{X}$, $\overline{\overline{X}} = \overline{X}$, and $X \subseteq Y \Longrightarrow \overline{X} \subseteq \overline{Y}$.

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If $\mathcal{F}$ is a closure system over A, then the mapping

$$X \mapsto \bigcap\{Y \in \mathcal{F} \mid X \subseteq Y\}$$

is a closure operator.

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c_1 $\quad x \leqslant \overline{x}$;

c_2 $\quad \overline{x} = \overline{\overline{x}}$;

c_3 $\quad$ if $x \leqslant y$, then $\overline{x} \leqslant \overline{y}$.

An element $x \in L$ fulfilling $\overline{x} = x$ is **closed** under this operator.

Proposition

The elements of a complete lattice L which are closed under a closure operator on L, form a complete lattice with respect to the order in L.

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Proposition

If F is a subset of a complete lattice L, and F is closed under arbitrary infima, then the mapping $x \mapsto \overline{x}$, where $\overline{x} = \bigwedge\{y \in F \mid x \leqslant y\}$, is a closure operator on L.

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C is a **partial closure operator** on S. If $C(X) = X$, then X is a **closed** set.

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Theorem

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Theorem

Every poset $(P, \leq)$ is isomorphic to a partial closure system on P, ordered by inclusion.

Let $(P, \leqslant)$ be a poset. A mapping $p \mapsto \overline{p}$ is a **closure operator on** P if the following conditions are satisfied for all $p, q \in P$:

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An element $p \in P$ is **closed** with respect to $\overline{\cdot}$ if $p = \overline{p}$.

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Let $(P, \leqslant)$ be a poset and $\mathcal{F}$ the set of all elements in P closed with respect to a given closure operator. Then, for every $p \in P$, $\bigwedge\{x \in \mathcal{F} \mid p \leqslant x\}$ exists and $\overline{p} = \bigwedge\{x \in \mathcal{F} \mid p \leqslant x\}$.

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Theorem

Let $(P, \leqslant)$ be a poset and $\mathcal{F} \subseteq P$ be such that for every $p \in P$ $\bigwedge\{x \in \mathcal{F} \mid p \leqslant x\}$ exists and belongs to $\mathcal{F}$. Then $p \mapsto \overline{p}$, where

$$\overline{p} := \bigwedge\{x \in \mathcal{F} \mid p \leqslant x\},$$

is a closure operator on P and $\mathcal{F}$ is the set of all closed elements with respect to $\overline{\cdot}$.

Lattice valued functions

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Lemma

If $\mu : A \to L$ is an L-valued function on A, then the collection μ_L of all cuts of μ is a closure system on A under the set-inclusion.

Lattice valued functions [2]

A function $f : \{0,1\}^n \to L$, where L is a complete lattice, is called a **lattice valued** (L**-valued**) **Boolean function**.

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For $x \in \{0,1\}$, and $w \in L$, we define a mapping $L \times \{0,1\}$ into L denoted by "$\cdot$", as follows:

$$w \cdot x := \begin{cases} w, & \text{if} \quad x = 1 \\ 0, & \text{if} \quad x = 0. \end{cases} \tag{1}$$

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Let $B = (\{0,1\}^n, \leq)$ be a Boolean lattice, L a complete lattice, $x = (x_1, \ldots, x_n) \in B$ and $w_1, \ldots, w_n \in L$. Further, let the binary function "$\cdot$" which maps $L \times \{0,1\}$ into L be defined by (1). Then the term

$$\bigvee_{i=1}^{n} (w_i \cdot x_i), \tag{2}$$

is a **linear combination** of elements $w_1, \ldots, w_n$ from L.

Lattice valued functions [2]

We say that a mapping $\mu : M \to L$ is a 0–$\vee$–**homomorphism**, if for all $x, y \in M$ (where the bottom elements of lattices M and L are 0_M and 0_L, respectively),

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Theorem

Let $B = (\{0,1\}^n, \leq)$ be a finite Boolean lattice and L an arbitrary complete lattice. Then an L-valued function $\mu : \{0,1\}^n \to L$ can be represented in the form

$$\mu(x) = \bigvee_{i=1}^{n} (w_i \cdot x_i)$$

for some elements $w_1, \ldots, w_n \in L$ if and only if μ as a mapping from B to L is a 0–$\vee$–homomorphism.

Lattice valued up-sets

As defined in [1], an L-valued function $\mu : P \to L$ on a poset $(P, \leq)$ is called a **lattice valued** (*L*-**valued**) **up-set**, or a **lattice valued** (*L*-**valued**) **semi-filter** on P if from $x \leq y$ it follows that $\mu(x) \leq \mu(y)$.

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Proposition

$\mu : P \to L$ is a lattice valued up-set on a poset $(P, \leq)$, if and only if for every $m \in L$, the cut μ_m is an up-set (semi-filter) on $(P, \leq)$.

Problem

Find necessary and sufficient conditions under which a lattice valued up-set $\mu : D \to L$ *on a finite distributive lattice* D *can be represented by the linear combination*

$$\mu(x) = \bigvee_{i=1}^{n} (w_i \cdot x_i)$$

over L *(*$x = (x_1, \ldots, x_n) \in \{0,1\}^n, w_1, \ldots, w_n \in L$*).*

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Indeed, if we denote the $\vee$-irreducible elements in D by $a_1, \ldots, a_n$, then each $x \in D$ can be uniquely represented as $x = \bigvee_{i \in I_x} a_i$, where $I_x = \{i \in \{1, \ldots, n\} \mid a_i \leqslant x\}$.

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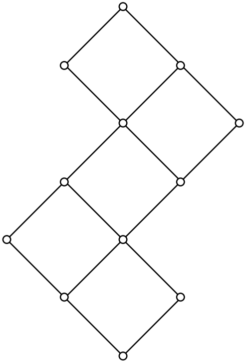
Also, x will be represented as an element of the Boolean algebra $\{0,1\}^n$ as $x = (x_1, \ldots, x_n)$, where $x_i = 1$ if and only if $i \in I_x$. This representation is unique up to isomorphism.

Example.



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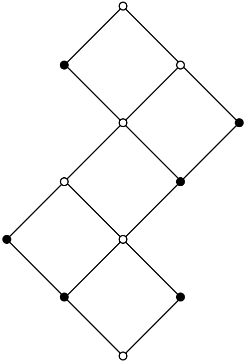

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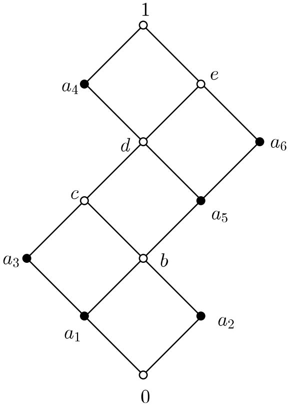

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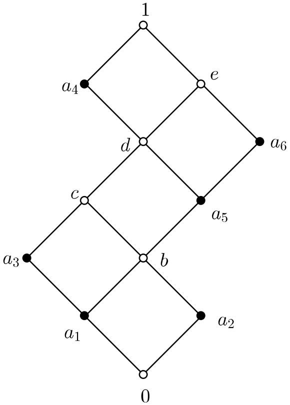

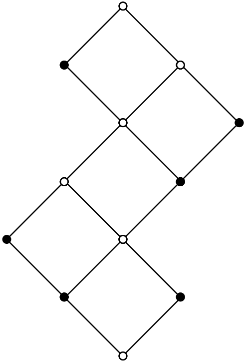

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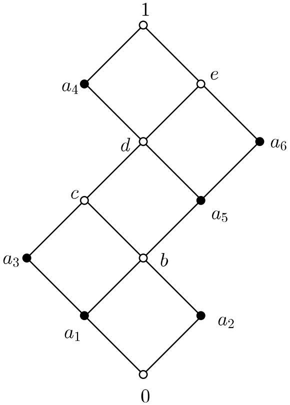

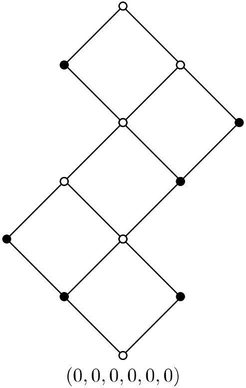

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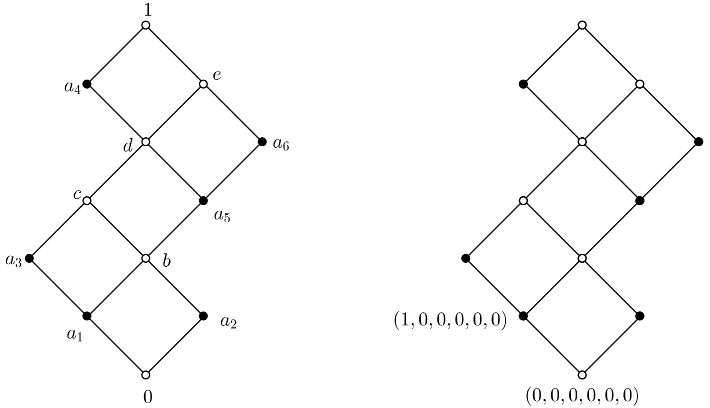

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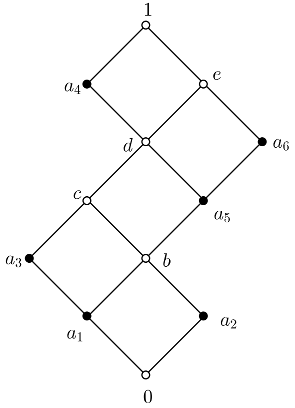
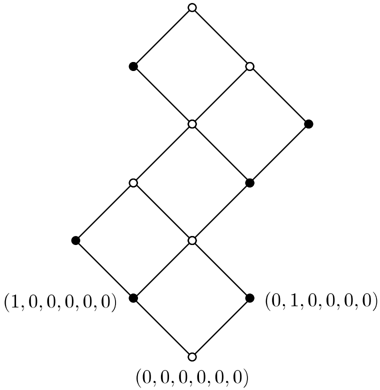
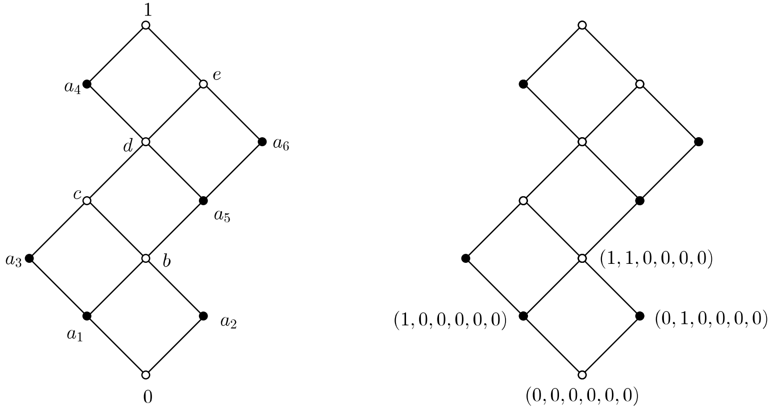
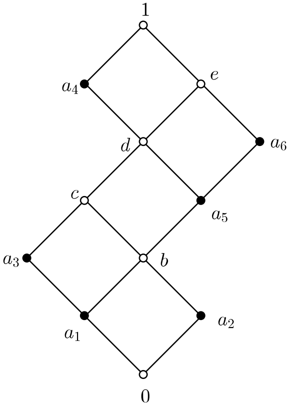


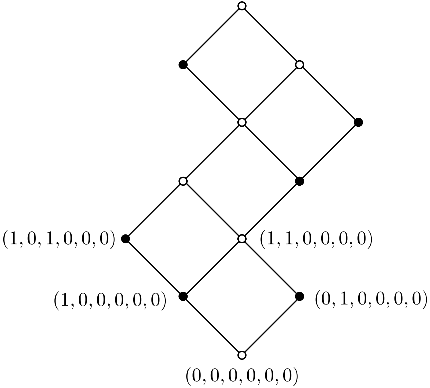


Example.

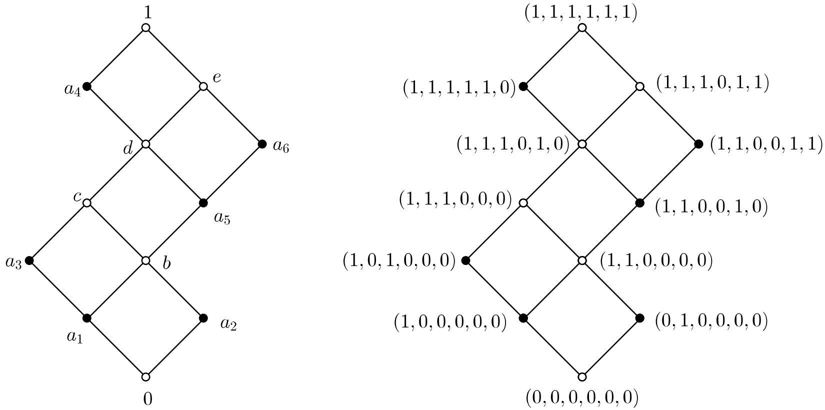


Example.





Example.


1
a4
e
d
a6
c
a5
a3
b
a1
a2
0
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Lemma

Let D be a distributive lattice and let $a \in D$ be a $\vee$-irreducible element. Then for each $x, y \in D$ we have

$$a \leqslant x \text{ or } a \leqslant y \text{ if and only if } a \leqslant x \vee y.$$

Distributive lattices

Lemma

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Corollary

Let D be a distributive lattice and $x, y \in D$. Then $I_x \cup I_y = I_{x \vee y}$.

Theorem

Let D be a finite distributive lattice with n $\vee$-irreducible elements represented as a sublattice of the Boolean algebra $\{0,1\}^n$. Let L be an arbitrary complete lattice and $\mu : D \to L$ be an L-valued up-set. Then μ can be represented in the form

$$\mu(x) = \bigvee_{i=1}^{n} (w_i \cdot x_i) = \bigvee_{i \in I_x} w_i$$

for some elements $w_1, \ldots, w_n \in L$ if and only if μ is a 0-$\vee$-homomorphism.

Theorem

Let $\mathcal{F}$ be a closure system of some up-sets on a finite distributive lattice D and for $x \in D$, define $\overline{x}$ by :

$$\overline{x} = \bigcap \{f \in \mathcal{F} \mid x \in f\}.$$

The following conditions are equivalent:
(i) for all $x, y \in D$

$$\text{from } \overline{x} \subseteq \overline{y} \text{ it follows that } \overline{x \vee y} = \overline{x}.$$

(ii) for all $x, y \in D$, $\overline{x \vee y} = \overline{x} \cap \overline{y}$.
(iii) There is a lattice L such that $\mathcal{F}$ is a family of cuts of an L-valued up-set on D which can be represented as a linear combination over L.

Non-distributive lattices

Non-distributive lattices

If L_1 is finite lattice whose $\vee$-irreducible elements are $a_1, \ldots, a_n$ then each element of L_1 can be uniquely represented as

$$x = \bigvee_{i \in I_x} a_i,$$

where $I_x = \{i \in \{1, \ldots, n\} \mid a_i \leqslant x\}$.

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Theorem

Let L_1 be a finite lattice, L be an arbitrary complete lattice and $\mu : L_1 \to L$ be an L-valued up-set. Now, if μ is a 0-$\vee$-homomorphism, then it can be represented in the form

$$\mu(x) = \bigvee_{i=1}^{n} (w_i \cdot x_i) = \bigvee_{i \in I_x} w_i$$

for some elements $w_1, \dots, w_n \in L$.

The opposite implication does not generally hold for finite non-distributive lattices, due to the following proposition.

Recall that an element a from a lattice L is said to be **co-distributive**, if for all $x, y \in L$

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$$a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y).$$

Proposition

If L is a finite lattice in which every $\vee$-irreducible element is co-distributive, then L is a distributive lattice.

Problems and further work

A **poset valued functions** are mappings from a nonempty set A into a poset $(P, \leq)$.

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Theorem

If $\mu : A \to P$ is a poset valued function on A, then the collection of cuts of μ is a partial closure (i.e., centralized) system on A under inclusion.

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This work is in progress.

References

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Thank you for your attention!