Aggregation on finite lattices

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- The main problem of the so called **Social choice theory** is the problem of aggregating individual preferences into a collective preference. A lattice theoretical approach to this problem was developed first in K. J. Arrow's book (**Social Choice and Individual Values, 1951**). Latter C. P. Chambers and A. D. Miller were working with the lattice of partitions of a finite set, using them to model individual preferences.

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- A main theorem of them was extended by Leclerc and Monjardet(2013) to every finite simple atomistic lattice having cardinality greater than two.

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Let L be a finite lattice and denote by J the set of its join-irreducible elements. For an element $x \in L$ we define $J(x) := \{j \in J \mid j \leq x\}$. Then

$$x = \bigvee J(x).$$

Let n be a fixed positive integer, and $I_n = \{1, 2, \ldots, n\}$. A **profile** π is an element $\pi = (x_1, x_2, \ldots, x_n)$ of L^n. We will use the notations $\pi(i) = x_i$ and $\pi_x = (x, x, \ldots, x)$ -for the constant profile belonging to x.
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(2) F is a **residual mapping** if it is a **meet homomorphism** (i.e. $F(\pi \wedge \pi') = F(\pi) \wedge F(\pi')$ for all $\pi, \pi' \in L^n$) such that $F(1, 1, \ldots, 1) = 1$.
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(3) is **Paretian** if for any $j \in J$ and $\pi \in L^n$, $N_j(\pi) = I_n$ implies $j \leq F(\pi)$. It is easy to see that this means that $x_1 \wedge \ldots \wedge x_n \leq F(x_1, \ldots, x_n)$, in other words, F is meet-dominating.

(4) F is **decisive** if $N_j(\pi) = N_j(\pi')$ yields $j \leq F(\pi) \Leftrightarrow j \leq F(\pi')$.

(5) F is **neutral monotone** if for all $j, j' \in J$, and all profiles $\pi, \pi' \in L^n$, $N_j(\pi) \subseteq N_{j'}(\pi')$ implies that if $j \leq F(\pi)$ then $j' \leq F(\pi')$.

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Theorem 1.

Let L be a finite simple atomistic lattice having cardinality greater than 2, and $F\colon L^n \to L$ a consensus function on L. The following conditions are then equivalent:

(F1) F is decisive and Paretian.

(F2) F is neutral monotone and is not F^0.

(F3) F is a meet homomorphism and $F(\pi) \geq \bigwedge_{i \in I_n} \pi(i)$ for all profiles π.

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At first, we present a natural generalization of this theorem.

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Let L be a finite tolerance simple lattice having cardinality at least 3, and $F: L^n \to L$. The following conditions are equivalent:

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A **tolerance** of a lattice L is a reflexive, symmetric relation on L compatible with the operations $\vee$ and $\wedge$. L is **congruence simple**, if it has only trivial congruences, namely $\triangle = \{(x,x) \mid x \in L\}$ and $\nabla = L \times L$, and L is **tolerance simple**, if it has only the tolerances $\triangle$ and ∇.

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A sketch of the proof

It can be checked that on any finite lattice with at least 3 elements (F1) $\Leftrightarrow$ (F2), (F3) $\Rightarrow$ (F4), and (F5) $\Rightarrow$ (F2).

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(F4) $\Rightarrow$ (F5): Here we need some technical details about the relation between the tolerances of a lattice L and its adjoint pairs of mappings.

Recall that a mapping $f \colon L \to L$ is increasing if $f(x) \geq x$, for all $x \in L$.

(the dual notion is an incresing mapping $f \colon L \to L$)

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Let L, M be finite lattices, with $F\colon L \to M$ and $G\colon M \to L$ mappings. F is **residual** if it is a meet homomorphism such that $F(1) = 1$, and G is **residuated** if G is a join homomorphism such that $G(0) = 0$. Every residual map $F\colon L \to M$ has a unique residuated mapping $G\colon M \to L$ associated with it. They are related by the equations

$$G(y) \leq x \iff y \leq F(x)\ .\ (*)$$

If $M = L$ then (G, F) is also called an **adjoint pair**. In this case $F\colon L \to L$ is increasing and $G\colon L \to L$ is decreasing.

Conversely, any pair of mappings such that $G\colon L \to L$ is decreasing (or $F\colon L \to L$ is increasing) which satisfies $(*)$ is an adjoint pair.

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Any adjoint pair of mappings defines a tolerance T_G on the lattice L by

$$T_G = \{(x, y) \in L^2 \mid G(x \vee y) \leq x \wedge y\}.$$

Conversely, any tolerance $T \subseteq L^2$ induces on a (finite) lattice L an adjoint pair of mappings as follows (see [6]):

$$G_T(x) := \bigwedge\{y \in L \mid (x, y) \in T\} \text{ and } F_T(x) := \bigvee\{y \in L \mid (x, y) \in T\}.$$

Now the main idea of the proof of (F4) $\Rightarrow$ (F5) is the following:
Suppose that $F\colon L^n \to L$ satisfies (F4), i.e. F is a residual map such that

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$$G(j) = (G_1(j), G_2(j), \ldots, G_n(j))$$

and the inequality $j \leq F(\pi_j)$ implies $G_i(j) \leq j$: $\forall i \in I_n$. Since each G_i is a join-homomorphism, we get

$G_i(x) = G_i(\bigvee J(x)) = \bigvee\{G_i(j) \mid j \leq x, j \in J\} \leq \bigvee\{j \in J \mid j \leq x\} = x$,
for all $x \in L$, and each i.

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Thus G_i is a decreasing residuated map, so it is associated with a unique tolerance on L. Because L is tolerance simple, it has only two such decreasing maps, namely the identity map and the zero map.
Let $M = \{i \in I_n \mid G_i(x) = x\}$. Then for $i \notin M$, $G_i(x) = 0$, for all $x \in L$. Hence

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By an easy computing it follows that:

$$F(\pi) = \bigwedge\{\pi(i) \mid i \in M\},$$

that is, F is an oligarchy.

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Let $L = (L, \wedge)$ be a meet semilattice with least element 0, $\{c_1, c_2, \ldots c_m\}$ a finite family of pairwise disjoint nonzero elements of L, and $I_m = \{1, 2, \ldots, m\}$. Denote by $D = \prod_{k \in I_m} [0, c_k]$ their direct product. For each $k \in I_m$, we define a mapping $\tau_k \colon [0, c_k] \to D$ by specifying that

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We say that L is an **internal subdirect product** of the semilattices $[0, c_k], k \in I_m$ if there exists a subdirect embedding Ψ of $(L, \wedge)$ into D such that for each $k \in I_m$ and every $x \leq c_k$ it is true that $\Psi(x) = \tau_k(x)$.

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Proposition 1.

Let L be a complete lattice and $S = \{c_k \mid k \in K\}$ a nonempty set of nonzero elements of L. The following conditions are then equivalent:

(i) S is a classification system of L.

(ii) The semilattice $(L, \wedge)$ is an internal subdirect product of the semilattices
$([0, c_k], \wedge)$, $k \in K$.

6. Aggregation and classification systems

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Let L be a finite lattice, $S = \{c_1, c_2, \ldots, c_m\}$ a classification system of L, and $F : L^n \to L$ a consensus function on L. We say that F is compatible with the system S if for each $c_k \in S$, and any profiles $\pi = (x_1, x_2, \ldots, x_n) \in L^n$, $\pi' = (y_1, y_2, \ldots, y_n) \in L^n$.

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(1) The **main observation** is that any meet homomorphism $F\colon L^n \to L$ which satisfies $F(\pi_x) \geq x$ for all $x \in L$ is S-compatible. Hence any residual map $F\colon L^n \to L$ induces a residual map F_k on each interval $[0, c_k]$. If this is tolerance simple, then the only induced maps on $[0, c_k]$ are the identity mapping and the constant mapping $F_k(x) = c_k$.

(2) We also recall that every join irreducible of L is in fact a join irreducible of some $[0, c_k]$. Since any join irreducible of $[0, c_k]$ must also be a join irreducible of L, we get $J(L) = \bigcup_{k \in I_m} J([0, c_k])$. Now, we obtain:

(2) We also recall that every join irreducible of L is in fact a join irreducible of some $[0, c_k]$. Since any join irreducible of $[0, c_k]$ must also be a join irreducible of L, we get $J(L) = \bigcup_{k \in I_m} J([0, c_k])$. Now, we obtain:

Theorem 3.

Let F be a consensus function on the fixed finite lattice L. Suppose $S = \{c_1, c_2, \ldots, c_m\}$ is a classification system for L, with each $[0, c_k]$ tolerance simple, having cardinality at least 3. The following conditions are then equivalent:

(P1) F is decisive, Paretian, and S-compatible.

(P2) F is neutral monotone and S-compatible, but $F \neq F^0$.

(P3) F is a meet homomorphism and $F(\pi) \geq \bigwedge_j \pi(j)$ for any profile π.

(P4) F is a residual map and $F(\pi_j) \geq j$ for every join irreducible j.

(P5) F is a *generalized oligarchy* in the sense that for every $c_k \in S$, the induced consensus function F_k defined on $[0, c_k]$ by $F_k(\pi \wedge \pi_{c_k}) = F(\pi) \wedge c_k$ is an oligarchy. In fact, there exist nonempty subsets $J_1, \ldots, J_m$ of I_m such that $F(\pi) = \left(\bigwedge_{j \in J_1} \pi(j) \wedge c_1\right) \vee \ldots \vee \left(\bigwedge_{j \in J_m} \pi(j) \wedge c_m\right)$.

- Blyth, T. S., Janowitz, M. F.: Residuation Theory, Pergamon (1972).
- Chambers, C. P., Miller, A. D.: Rules for aggregating information, Social Choice and Welfare **36**, 75–82 (2011).
- Ganter, B. and Wille, R.: *Formal concept analysis: Mathematical foundations*, Springer, Berlin-Heidelberg, 1999.
- Ganter, B., Körei, A., Radeleczki, S.: Extent partitions and context extensions, Math. Slovaca **63 (4)**, 693–706 (2013).
- D. Hobby and R. McKenzie *The structure of finite algebras*, AMS Contemporary Mathemaics 76 **76** Providence-Rhode Island, (1988).
- Janowitz, M. F.: Decreasing Baer semigroups, Glasgow Math. J. **10**, 46–51 (1969).

- Janowitz, M. F.: Tolerances and congruences on lattices, Czechoslovak Math. J. **36**, 108–115 (1986).
- Leclerc, B., Monjardet, B.: Aggregation and Residuation, Order **30**, 261–268 (2013)
- Monjardet, B.: Arrowian characterization of latticial federation consensus functions, Math. Social Sciences **20**, 51–71 (1990).
- Radeleczki, S.: Classification systems and their lattice, Discussiones Mathematicae, Gen. Algebra and Applications, **22**, 167–181 (2002).

Thank You for your kind attention !

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