Report on Centralizing Monoids on E_3

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Joint work with Ivo G. Rosenberg

Ivo Rosenberg asked me to convey

BEST REGARDS

to

PARTICIPANTS of the Summer School !!

1. Centralizing Monoids

Notation

A : non-empty set

$\mathcal{O}_A^{(n)} (= A^{A^n})$: the set of n-variable functions on A

$$\mathcal{O}_A = \bigcup_{n=1}^{\infty} \mathcal{O}_A^{(n)}$$

Commutation

Definition

For $f \in \mathcal{O}_A^{(m)}$ and $g \in \mathcal{O}_A^{(n)}$,
f and g commute (expressed as $f \perp g$)
if the following holds for all $m \times n$ matrix $M = (x_{ij})$ over A

$$\begin{aligned} & f(g(x_{11}, \ldots, x_{1n}), \ldots, g(x_{m1}, \ldots, x_{mn})) \\ = \ & g(f(x_{11}, \ldots, x_{m1}), \ldots, f(x_{1n}, \ldots, x_{mn})) \end{aligned}$$

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$$\begin{array}{cccc|c} x_{11} & x_{12} & \cdots & x_{1n} & g(\ldots, x_{1j}, \ldots) \\ x_{21} & x_{22} & \cdots & x_{2n} & g(\ldots, x_{2j}, \ldots) \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} & g(\ldots, x_{m1}, \ldots) \\ \hline f(\ldots, x_{i1}, \ldots) & f(\ldots, x_{i2}, \ldots) & \cdots & f(\ldots, x_{in}, \ldots) & f(g, \ldots) = g(f, \ldots) \end{array}$$

Example

- $f \in \mathcal{O}_A^{(m)}$: constant function
- $g \in \mathcal{O}_A^{(n)}$: idempotent function

$\Longrightarrow$

f and g commute, i.e. , $f \perp g$

(Note: g is *idempotent* if $g(x, \ldots, x) = x$ for $\forall\, x \in A$.)

Centralizer

Definition

For $F \subseteq \mathcal{O}_A$ define F^* by

$$F^* \;=\; \{ \, g \in \mathcal{O}_A \mid g \perp f \ \text{ for all } \ f \in F \, \}$$

F^* is called the centralizer of F.

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F^* is called the centralizer of F.

Note: A centralizer is always a clone.

Basic properties of $*$ - operator:

For any $F,\ G \subseteq \mathcal{O}_A$

(i) $F \subseteq G \implies F^* \supseteq G^*$

(ii) $F \subseteq F^{**}$

(iii) $F^{***} = F^*$

Centralizing Monoids

Now we define "centralizing monoids"

which concern commutation of

a unary function and an $n\,(> 0)$-ary function.

Commutation for a unary function :

For $f \in \mathcal{O}_A^{(1)}$ and $g \in \mathcal{O}_A^{(n)}$,
f and g commute ($f \perp g$)
if the following holds for all $(x_1, \ldots, x_n) \in A^n$

$$f\,(\,g\,(x_1, \ldots, x_n)\,) \;=\; g\,(\,f\,(x_1), \ldots, f\,(x_n)\,)$$

Lemma

For $M \subseteq \mathcal{O}_A^{(1)}$, the following conditions are equivalent.

(1) $M = M^{**} \cap \mathcal{O}_A^{(1)}$

(2) $\exists\, F \subseteq \mathcal{O}_A$, $M = F^* \cap \mathcal{O}_A^{(1)}$

(3) $\exists\, \mathcal{A} = (A;\ F)$: algebra, $M = \mathrm{End}\,(\mathcal{A})$

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(i.e., M is the unary part of some centralizer)

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(i.e., M is the unary part of some centralizer)

(3) $\exists\, \mathcal{A} = (A; F)$: algebra, $\quad M = \mathrm{End}\,(\mathcal{A})$

Definition

For $M \subseteq \mathcal{O}_A^{(1)}$, M is a **centralizing monoid** if M satisfies the above conditions.

Witness

Definition

When $M = F^* \cap \mathcal{O}_A^{(1)}$ for $F \subseteq \mathcal{O}_A$,

we say

F is a witness of a centralizing monoid M.

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When $M = F^* \cap \mathcal{O}_A^{(1)}$ for $F \subseteq \mathcal{O}_A$,
we say
F is a witness of a centralizing monoid M.

Notation

Denote by $M(F)$ the centralizing monoid M
which has F as its witness.

Theorem

For every centralizing monoid M there exists a finite subset of $\mathcal{O}_A$ which is a witness of M,

that is,

" every centralizing monoid M has a finite witness. "

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Proof. Suppose $M \neq \mathcal{O}_A^{(1)}$. (The case $M = \mathcal{O}_A^{(1)}$ is trivial.)
Take arbitrary witness $F \,(\subseteq \mathcal{O}_A)$ for M.
For each $f \in \mathcal{O}_A^{(1)} \setminus M$ there exists $u \in F$ such that $f \not\perp u$.
We pick one such $u \,(= u_f)$ for each f and let

$$T = \{\, u_f \mid f \in \mathcal{O}_A^{(1)} \setminus M \,\}.$$

Then, T is clearly a witness for M and, moreover, T is finite because $\mathcal{O}_A^{(1)}$ is finite. □

Now
we shall focus on
maximal centralizing monoids,

which are related to
minimal clones !!

2. Maximal Centralizing Monoids

Definition

A centralizing monoid M is maximal
if $\mathcal{O}_A^{(1)}$ is the only centralizing monoid properly containing M.

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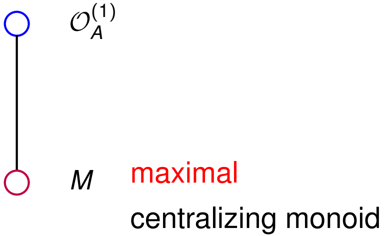

Lemma

For any maximal centralizing monoid M, there exists $u\,(\in \mathcal{O}_A)$ such that

$$M = M(u),$$

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" every maximal centralizing monoid has a singleton witness. "

(**Proof** $\quad M\,(S_1) \cap M\,(S_2) = M\,(S_1 \cup S_2)$)

Theorem

For any maximal centralizing monoid M, there exists a minimal function $f \, (\in \mathcal{O}_A)$ such that

$$M = M(f) \, ,$$

that is,

" every maximal centralizing monoid has a witness which is a minimal function. "

Proof

- Since a maximal centralizing monoid has a singleton witness, there exists $g \in \mathcal{O}_A$ such that $M = M(g)$.
- Every non-trivial clone C (i.e., $C \neq \mathcal{J}_A$) contains a minimal clone. Hence, there exists $f \in \mathcal{O}_A$ which satisfies the following.

  (i) $\langle f \rangle$ is a minimal clone.
  (ii) $\langle f \rangle \subseteq \langle g \rangle \quad (\Leftrightarrow \ f \in \langle g \rangle)$

- In general, for any $u, v, w \in \mathcal{O}_A$,

$$u \in \langle v \rangle \ \text{and} \ v \perp w \quad \Longrightarrow \quad u \perp w$$

  As a corollary, $u \in \langle v \rangle \quad \Longrightarrow \quad v^* \subseteq u^*$

- Hence, for f and g given above, we have $g^* \subseteq f^*$.

Proof (cont.)

- It follows that

$$M(g) = g^* \cap \mathcal{O}_A^{(1)} \subseteq f^* \cap \mathcal{O}_A^{(1)} = M(f)$$

- Since $M(g)$ is a maximal centralizing monoid, by assumption, we have either

$$M(g) = M(f) \quad (\Rightarrow M = M(f))$$

or

$$M(f) = \mathcal{O}_A^{(1)}.$$

- However, it is known that $(\mathcal{O}_A^{(1)})^* = J_A$ and, hence, $M(f) \neq \mathcal{O}_A^{(1)}$ for a minimal function f, and so

$$M = M(f)$$

completing the proof. □

From here, we consider only the case where

$$A = E_3 = \{0, 1, 2\}$$

Denote $\mathcal{O}_A^{(n)}$ and $\mathcal{O}_A$ by $\mathcal{O}_3^{(n)}$ and $\mathcal{O}_3$, respectively.

3. Review: Minimal Clones on E_3

Next,

we review

the minimal clones on E_3

The list of all minimal clones on E_3 was obtained by B. Csákány (1983).

They are generated by

- Unary functions
- Binary idempotent functions
- Ternary majority functions
- Semiprojections

Number of minimal clones on E_3

Unary functions	:	13	(4)
Binary idempotent functions	:	48	(12)
Ternary majority functions	:	7	(3)
Ternary semiprojections	:	16	(5)
Total	:	84	(24)

4. Maximal Centralizing Monoids on $\{0, 1, 2\}$

Using the result of B. Csákány,
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we determined all maximal centralizing monoids on E_3.

Namely,

for each minimal function $f \in \mathcal{O}_3^{(1)}$, let $\{ f \}$ be a witness and determine a centralizing monoid $M(f)$.
Some of such centralizing monoids are maximal, while some are not maximal.

Example

To determine centralizing monoids having majority functions as their witnesses, the following properties, for example, are useful to reduce the amount of work.

Let $m(x, y, z)$ be a majority function given by

$m(x, y, z) =$

$x\backslash y$	0	1	2
0	0	0	0
1	0	1	d
2	0	f	2

$z = 0$

	0	1	2
0	0	1	b
1	1	1	1
2	e	1	2

$z = 1$

	0	1	2
0	0	a	2
1	c	1	2
2	2	2	2

$z = 2$

For each unary function $s \in \mathcal{O}_3^{(1)}$ we can describe a condition for which $s \perp m$ is satisfied.

For example:

- $\forall\, c \in \mathcal{O}_3^{(1)}$: constant, $c \perp m$
- ⟨ Permutation ⟩ Let $s = (0\ 1)$. Then

$$s \perp m \iff \begin{array}{l} (a, c) \in \{(0,1),\ (1,0),\ (2,2)\} \\ (b, d) \in \{(0,1),\ (1,0),\ (2,2)\} \\ (e, f) \in \{(0,1),\ (1,0),\ (2,2)\} \end{array}$$

- ⟨ 2-valued function ⟩ Let $s \in \mathcal{O}_3^{(1)}$ satisfy $s(0) = s(1) = 0,\ s(2) = 1$. Then

$$s \perp m \iff a, b, c, d, e, f \in \{0, 1\}$$

Proposition

There are 10 maximal centralizing monoids on E_3.

More precisely:

- there are 3 maximal centralizing monoids, each of which has a unary constant function as its witness.
- there are 7 maximal centralizing monoids, each of which has a ternary majority function which generates a minimal clone as its witness.

Minimal functions which, as witnesses, correspond to maximal centralizing monoids are:

Constant functions

$$\begin{aligned} c_0(x) &= 0 \\ c_1(x) &= 1 \\ c_2(x) &= 2 \end{aligned}$$

Majority functions

(showing the values for $|\{x, y, z\}| = 3$)

$$m_0(x, y, z) = 0 \quad \text{if} \quad |\{x, y, z\}| = 3$$

$$m_{364}(x, y, z) = 1 \quad \text{if} \quad |\{x, y, z\}| = 3$$

$$m_{728}(x, y, z) = 2 \quad \text{if} \quad |\{x, y, z\}| = 3$$

$$m_{109}(x, y, z) = \begin{cases} 0 & \text{if} \quad (x, y, z) \in \sigma \\ 1 & \text{if} \quad (x, y, z) \in \tau \end{cases}$$

$$m_{473}(x, y, z) = \begin{cases} 1 & \text{if} \quad (x, y, z) \in \sigma \\ 2 & \text{if} \quad (x, y, z) \in \tau \end{cases}$$

$$m_{510}(x, y, z) = \begin{cases} 2 & \text{if} \quad (x, y, z) \in \sigma \\ 0 & \text{if} \quad (x, y, z) \in \tau \end{cases}$$

$$m_{624}(x, y, z) = y \quad \text{if} \quad |\{x, y, z\}| = 3$$

where $\sigma = \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\}$

and $\tau = \{(0, 2, 1), (1, 0, 2), (2, 1, 0)\}$

Number of unary functions in maximal centralizing monoids on $\{0, 1, 2\}$:

$M(c_0)$	9
$M(c_1)$	9
$M(c_2)$	9
$M(m_0)$	17
$M(m_{364})$	17
$M(m_{728})$	17
$M(m_{109})$	11
$M(m_{473})$	11
$M(m_{510})$	11
$M(m_{624})$	9

Note: $|\mathcal{O}_3^{(1)}| = 27$

Remark :

There exist other minimal functions which serve as witnesses of maximal centralizing monoids.

They are:

Binary function: b_{624}

and

Semiprojections: p_{76}, p_{684}, p_{332} and p_{624}

However, the centralizing monoids having them as witnesses all coincide with already known centralizing monoids.

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Namely,

Binary function: $M(b_{624}) = M(m_{624})$

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Namely,

Binary function: $M(b_{624}) = M(m_{624})$

and

Semiprojections:

$$\begin{aligned} M(p_{76}) &= M(m_{473}) \\ M(p_{684}) &= M(m_{510}) \\ M(p_{332}) &= M(m_{109}) \\ M(p_{624}) &= M(m_{624}) \end{aligned}$$

(1) m_{624} vs b_{624}

$m_{624}(x, y, z) =$

$x \backslash y$	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

$z = 0$

	0	1	2
0	0	1	1
1	1	1	1
2	1	1	2

$z = 1$

	0	1	2
0	0	2	2
1	2	1	2
2	2	2	2

$z = 2$

$b_{624}(x, y) =$

$x \backslash y$	0	1	2
0	0	2	1
1	2	1	0
2	1	0	2

(2) m_{473} vs p_{76}

$m_{473}(x, y, z) =$

$x \backslash y$	0	1	2
0	0	0	0
1	0	1	1
2	0	2	2

$z = 0$

	0	1	2
0	0	1	2
1	1	1	1
2	1	1	2

$z = 1$

	0	1	2
0	0	1	2
1	2	1	2
2	2	2	2

$z = 2$

$p_{76}(x, y, z) =$

$x \backslash y$	0	1	2
0	0	0	0
1	1	1	2
2	2	1	2

$z = 0$

	0	1	2
0	0	0	0
1	1	1	1
2	1	2	2

$z = 1$

	0	1	2
0	0	0	0
1	2	1	1
2	2	2	2

$z = 2$

(3) m_{510} vs p_{684}

$m_{510}(x, y, z) =$

$x\backslash y$	0	1	2
0	0	0	0
1	0	1	2
2	0	0	2

$z = 0$

	0	1	2
0	0	1	0
1	1	1	1
2	2	1	2

$z = 1$

	0	1	2
0	0	2	2
1	0	1	2
2	2	2	2

$z = 2$

$p_{684}(x, y, z) =$

$x\backslash y$	0	1	2
0	0	0	0
1	1	1	1
2	2	0	2

$z = 0$

	0	1	2
0	0	0	2
1	1	1	1
2	0	2	2

$z = 1$

	0	1	2
0	0	2	0
1	1	1	1
2	2	2	2

$z = 2$

(4) m_{109} vs p_{332}

$m_{109}(x, y, z) =$

$x \backslash y$	0	1	2
0	0	0	0
1	0	1	0
2	0	1	2

$z = 0$

	0	1	2
0	0	1	1
1	1	1	1
2	0	1	2

$z = 1$

	0	1	2
0	0	0	2
1	1	1	2
2	2	2	2

$z = 2$

$p_{332}(x, y, z) =$

$x \backslash y$	0	1	2
0	0	0	0
1	1	1	0
2	2	2	2

$z = 0$

	0	1	2
0	0	0	1
1	1	1	1
2	2	2	2

$z = 1$

	0	1	2
0	0	1	0
1	0	1	1
2	2	2	2

$z = 2$

(5) m_{624} vs p_{624}

$m_{624}(x, y, z) =$

$x \backslash y$	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

$z = 0$

	0	1	2
0	0	1	1
1	1	1	1
2	1	1	2

$z = 1$

	0	1	2
0	0	2	2
1	2	1	2
2	2	2	2

$z = 2$

$p_{624}(x, y, z) =$

$x \backslash y$	0	1	2
0	0	0	0
1	1	1	0
2	2	0	2

$z = 0$

	0	1	2
0	0	0	1
1	1	1	1
2	1	2	2

$z = 1$

	0	1	2
0	0	2	0
1	2	1	1
2	2	2	2

$z = 2$

5. Generalizing the Results from E_3 to E_k $(k > 3)$

On E_3, we have shown the following:

1. $\forall\, c\ (\in \mathcal{O}_3^{(1)})$: constant function,
   $M(c) = $ maximal centralizing monoid
2. $\forall\, m\ (\in \mathcal{O}_3^{(3)})$: minimal majority function,
   $M(m) = $ maximal centralizing monoid

Conversely,

3. $\forall\, M$: maximal centralizing monoid,
   $\exists\, f\ (\in \mathcal{O}_3)$: constant function or minimal majority function such that $M = M(f)$.

Question :

Can we generalize these results from E_3 to E_k $(k > 3)$?

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Answer :

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Answer :

1. $\forall\ c\ (\in \mathcal{O}_k^{(1)})$: constant function,
   $M(c) \ = \ $ maximal centralizing monoid YES
2. $\forall\ m\ (\in \mathcal{O}_k^{(3)})$: minimal majority function,
   $M(m) \ = \ $ maximal centralizing monoid OPEN
3. $\forall\ M$: maximal centralizing monoid,
   $\exists\ f\ (\in \mathcal{O}_k)$: constant function **or** minimal majority function such that $M \ = \ M(f)$. OPEN

(1) Case of Constant Functions

Theorem (M & R; M. Goldstern)

For any $k > 1$ and any constant function c on E_k, $M(c)$ is a maximal centralizing monoid.

Proof

We assume, w.l.o.g, $c = c_0$, the constant taking value 0.

Lemma 1

$$M(c_0) = (\mathrm{Pol}(0))^{(1)}$$

Lemma 2

$$(\mathrm{CONST})^* = \mathrm{IDEMP}$$

Lemma 3

For $f \in \mathcal{O}_k$, if $f \in (\mathrm{Pol}(0)^{(1)})^* \cap \mathrm{IDEMP}$ then f is conservative.

(f : *conservative* $\overset{\text{def}}{\iff} \forall D \subseteq E_k,\ f(D, \ldots, D) \subseteq D$)

Lemma 4

$(\mathrm{Pol}(0)^{(1)})^* \cap \mathrm{IDEMP} = \mathcal{J}_k$ (= the clone of projections)

$\langle\langle$ This lemma requires a bit of work. $\rangle\rangle$

Proof of Theorem Let M be a monoid containing $M(c_0) \cup \{u\}$ for some $u \in \mathcal{O}_k^{(1)} \setminus M(c_0)$. Since $M(c_0) = \mathrm{Pol}(0)^{(1)}$, u maps 0 to some $a \neq 0$. Then M must contain all constant functions. Hence we have

$$M \supset M(c_0) \cup \mathrm{CONST}.$$

It follows that

$$M^* \subseteq M(c_0)^* \cap (\mathrm{CONST})^*,$$

which implies, by Lemmas 1 and 2, that

$$M^* \subseteq (\mathrm{Pol}(0)^{(1)})^* \cap \mathrm{IDEMP}.$$

Hence, it follows by Lemma 4 that

$$M^* = \mathcal{J}_k.$$

By applying $*$ to both sides of $M^* = \mathcal{J}_k$, we obtain

$$M^{**} = \mathcal{J}_k^* \,(= \mathcal{O}_k).$$

Hence

$$M^{**} \cap \mathcal{O}_k^{(1)} = \mathcal{O}_k^{(1)}.$$

Therefore, if M is a centralizing monoid then, by definition,

$$M \,(= M^{**} \cap \mathcal{O}_k^{(1)}) = \mathcal{O}_k^{(1)}.$$

This concludes that $M(c_0)$ is a maximal centralizing monoid. $\square$

(2) Case of Minimal Majority Functions

Is it true that,

for every majority function m, if m is minimal then $M(m)$ is a maximal centralizing monoid ?

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Still open.

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Still open.

It may happen that:

- YES if m is conservative.
- NO if m is not conservative.

(2) Case of Minimal Majority Functions

Is it true that,

for every majority function m, if m is minimal then $M(m)$ is a maximal centralizing monoid ?

Still open.

It may happen that:

- YES if m is conservative.
- NO if m is not conservative.

(f : conservative $\overset{\text{def}}{\Longleftrightarrow} \forall D \subseteq E_k,\ f(D, \ldots, D) \subseteq D$)

Remark:

Non-conservative minimal majority functions on E_4

(due to T. Waldhauser)

(x, y, z)	M_1	M_2	M_3
(0, 1, 2)	3	3	2
(1, 2, 0)	3	1	2
(2, 0, 1)	3	2	2
(1, 0, 2)	3	1	3
(0, 2, 1)	3	3	3
(2, 1, 0)	3	2	3
$\{0, 1, 3\}$	3	3	3
$\{0, 2, 3\}$	3	3	3
(3, 1, 2)	3	3	2
(1, 2, 3)	3	1	2
(2, 3, 1)	3	2	2
(1, 3, 2)	3	1	3
(3, 2, 1)	3	3	3
(2, 1, 3)	3	2	3

Size of $M(M_i)$ for $i = 1, 2, 3$

$$\begin{aligned} |\, M(M_1) \,| &= 64 \\ |\, M(M_2) \,| &= 24 \\ |\, M(M_3) \,| &= 31 \end{aligned}$$

Note that $|\, \mathcal{O}_4^{(1)} \,| = 256$

For the Future

There are two possibilities.

The DARK Future

The DARK Future

- There is no unified theory which captures the situation for all finite k (> 2),

  or

- there is a unified theory which, however, is too complicated and far beyond our ability to describe it.

The BRIGHT Future

The BRIGHT Future

- There exists a nice unified theory which captures the situation for all finite k (> 2),

  and

- we are able to discover this theory and report it

The BRIGHT Future

- There exists a nice unified theory which captures the situation for all finite k (> 2),

  and

- we are able to discover this theory and report it at

  the **next** **Summer School** in **2015**.

Thank you

for your attention !