On quasivariety of pseudo BCI-algebras

Petr Emanovský, Jan Kühr

Department of Algebra and Geometry
Faculty of Science
Palacký University at Olomouc
Olomouc, Czech Republic
petr.emanovsky@upol.cz, jan.kuhr@upol.cz

A pseudo BCI-algebra is an algebra $(A, \to, \rightsquigarrow, 1)$, where $\to$ and $\rightsquigarrow$ are binary operations on A and 1 is an element of A, stisfying the following axioms, for all $x, y, z \in A$:

(A1) $(x \to y) \rightsquigarrow ((y \to z) \rightsquigarrow (x \to z)) = 1$,

(A2) $(x \rightsquigarrow y) \to ((y \rightsquigarrow z) \to (x \rightsquigarrow z)) = 1$,

(A3) $1 \to x = x$,

(A4) $1 \rightsquigarrow x = x$,

(A5) if $x \to y = 1$ and $y \to x = 1$ then $x = y$.

(W. A. Dudek, Y. B. Yun, 2008)

- The relation $\leq = \{(x, y) \in A^2 \mid x \to y = 1\}$ is a partial order on A with 1 as a maximal element.
- If 1 is the greatest element of A then $(A, \to, \rightsquigarrow, 1)$ is a pseudo BCK-algebra (G. Georgescu, A. Iorgulescu, 2001).
- If the operations $\to$ and $\rightsquigarrow$ coincide then $(A, \to, 1)$ is a BCI-algebra (K. Iseki, 1980).
- Pseudo BCI-algebras form a proper quasi-variety.

A pseudo BCI-algebra is an algebra $(A, \to, \rightsquigarrow, 1)$, where $\to$ and $\rightsquigarrow$ are binary operations on A and 1 is an element of A, stisfying the following axioms, for all $x, y, z \in A$:

(A1) $(x \to y) \rightsquigarrow ((y \to z) \rightsquigarrow (x \to z)) = 1$,

(A2) $(x \rightsquigarrow y) \to ((y \rightsquigarrow z) \to (x \rightsquigarrow z)) = 1$,

(A3) $1 \to x = x$,

(A4) $1 \rightsquigarrow x = x$,

(A5) if $x \to y = 1$ and $y \to x = 1$ then $x = y$.

(W. A. Dudek, Y. B. Yun, 2008)

- The relation $\leq = \{(x, y) \in A^2 \mid x \to y = 1\}$ is a partial order on A with 1 as a maximal element.
- If 1 is the greatest element of A then $(A, \to, \rightsquigarrow, 1)$ is a pseudo BCK-algebra (G. Georgescu, A. Iorgulescu, 2001).
- If the operations $\to$ and $\rightsquigarrow$ coincide then $(A, \to, 1)$ is a BCI-algebra (K. Iseki, 1980).
- Pseudo BCI-algebras form a proper quasi-variety.

A pseudo BCI-algebra is an algebra $(A, \to, \rightsquigarrow, 1)$, where $\to$ and $\rightsquigarrow$ are binary operations on A and 1 is an element of A, stisfying the following axioms, for all $x, y, z \in A$:

(A1) $(x \to y) \rightsquigarrow ((y \to z) \rightsquigarrow (x \to z)) = 1$,

(A2) $(x \rightsquigarrow y) \to ((y \rightsquigarrow z) \to (x \rightsquigarrow z)) = 1$,

(A3) $1 \to x = x$,

(A4) $1 \rightsquigarrow x = x$,

(A5) if $x \to y = 1$ and $y \to x = 1$ then $x = y$.

(W. A. Dudek, Y. B. Yun, 2008)

- The relation $\leq = \{(x, y) \in A^2 \mid x \to y = 1\}$ is a partial order on A with 1 as a maximal element.
- If 1 is the greatest element of A then $(A, \to, \rightsquigarrow, 1)$ is a pseudo BCK-algebra (G. Georgescu, A. Iorgulescu, 2001).
- If the operations $\to$ and $\rightsquigarrow$ coincide then $(A, \to, 1)$ is a BCI-algebra (K. Iseki, 1980).
- Pseudo BCI-algebras form a proper quasi-variety.

A pseudo BCI-algebra is an algebra $(A, \to, \rightsquigarrow, 1)$, where $\to$ and $\rightsquigarrow$ are binary operations on A and 1 is an element of A, stisfying the following axioms, for all $x, y, z \in A$:

(A1) $(x \to y) \rightsquigarrow ((y \to z) \rightsquigarrow (x \to z)) = 1$,

(A2) $(x \rightsquigarrow y) \to ((y \rightsquigarrow z) \to (x \rightsquigarrow z)) = 1$,

(A3) $1 \to x = x$,

(A4) $1 \rightsquigarrow x = x$,

(A5) if $x \to y = 1$ and $y \to x = 1$ then $x = y$.

(W. A. Dudek, Y. B. Yun, 2008)

- The relation $\leq = \{(x, y) \in A^2 \mid x \to y = 1\}$ is a partial order on A with 1 as a maximal element.
- If 1 is the greatest element of A then $(A, \to, \rightsquigarrow, 1)$ is a pseudo BCK-algebra (G. Georgescu, A. Iorgulescu, 2001).
- If the operations $\to$ and $\rightsquigarrow$ coincide then $(A, \to, 1)$ is a BCI-algebra (K. Iseki, 1980).
- Pseudo BCI-algebras form a proper quasi-variety.

A pseudo BCI-algebra is an algebra $(A, \to, \rightsquigarrow, 1)$, where $\to$ and $\rightsquigarrow$ are binary operations on A and 1 is an element of A, stisfying the following axioms, for all $x, y, z \in A$:

(A1) $(x \to y) \rightsquigarrow ((y \to z) \rightsquigarrow (x \to z)) = 1$,

(A2) $(x \rightsquigarrow y) \to ((y \rightsquigarrow z) \to (x \rightsquigarrow z)) = 1$,

(A3) $1 \to x = x$,

(A4) $1 \rightsquigarrow x = x$,

(A5) if $x \to y = 1$ and $y \to x = 1$ then $x = y$.

(W. A. Dudek, Y. B. Yun, 2008)

- The relation $\leq = \{(x, y) \in A^2 \mid x \to y = 1\}$ is a partial order on A with 1 as a maximal element.
- If 1 is the greatest element of A then $(A, \to, \rightsquigarrow, 1)$ is a pseudo BCK-algebra (G. Georgescu, A. Iorgulescu, 2001).
- If the operations $\to$ and $\rightsquigarrow$ coincide then $(A, \to, 1)$ is a BCI-algebra (K. Iseki, 1980).
- Pseudo BCI-algebras form a proper quasi-variety.

In any pseudo BCI-algebra $(A, \to, \rightsquigarrow, 1)$ hold for all $x, y, z \in A$:

1. $x \to x = 1$, $x \rightsquigarrow x = 1$,
2. $x \rightsquigarrow ((x \to y) \rightsquigarrow y) = 1$, $x \to ((x \rightsquigarrow y) \to y) = 1$,
3. $x \to y = 1$ iff $x \rightsquigarrow y = 1$,
4. $x \leq y$ implies $y \to z \leq x \to z$ and $y \rightsquigarrow z \leq x \rightsquigarrow z$,
5. $x \leq y$ implies $z \to x \leq z \to y$ and $z \rightsquigarrow x \leq z \rightsquigarrow y$,
6. $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$,
7. $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$,
8. $x \to y \leq (y \to x) \to 1$, $x \rightsquigarrow y \leq (y \rightsquigarrow x) \rightsquigarrow 1$,
9. $x \to 1 = x \rightsquigarrow 1$,
10. $(x \to y) \to 1 = (x \to 1) \rightsquigarrow (y \to 1)$,
    $(x \rightsquigarrow y) \rightsquigarrow 1 = (x \rightsquigarrow 1) \to (y \rightsquigarrow 1)$,
11. $((x \to y) \rightsquigarrow y) \to y = x \to y$,
    $((x \rightsquigarrow y) \to y) \rightsquigarrow y = x \rightsquigarrow y$.

Important subalgebras of pseudo-BCI-algebra $(A, \rightarrow, \rightsquigarrow, 1)$:

- Integral part of A ... $I_A = \{a \in A \mid a \leq 1\}$ - 1 is the top element of I_A, i.e. I_A is a pseudo BCK-algebra.

  $x \in I_A$ iff $((x \rightarrow 1) \rightarrow 1) \rightarrow x = x$

- Group part of A ... $G_A = \{a \rightarrow 1 \mid a \in A\}$

  $x \in G_A$ iff $((x \rightarrow 1) \rightarrow 1) = x$

Theorem

$(G_A, \cdot)$ where $x \cdot y = (x \rightarrow 1) \rightsquigarrow y = (y \rightsquigarrow 1) \rightarrow x$ is a group in which 1 is the identity and $x^{-1} = x \rightarrow 1 = x \rightsquigarrow 1$ is the inverse of $x \in G_A$. The original operations $\rightarrow$ and $\rightsquigarrow$ on G_A are retrieved from $\cdot$ by $x \rightarrow y = y \cdot x^{-1}$ and $x \rightsquigarrow y = x^{-1} \cdot y$.

Proposition

For every $a \in A$, $(a \rightarrow 1) \rightarrow 1$ is the only element $g \in G_A$ with $a \leq g$ and G_A is the set of all maximal elements of A.

Important subalgebras of pseudo-BCI-algebra $(A, \to, \rightsquigarrow, 1)$:

- Integral part of A ... $I_A = \{a \in A \mid a \leq 1\}$ - 1 is the top element of I_A, i.e. I_A is a pseudo BCK-algebra.

  $x \in I_A$ iff $((x \to 1) \to 1) \to x = x$

- Group part of A ... $G_A = \{a \to 1 \mid a \in A\}$

  $x \in G_A$ iff $((x \to 1) \to 1) = x$

Theorem

$(G_A, \cdot)$ where $x \cdot y = (x \to 1) \rightsquigarrow y = (y \rightsquigarrow 1) \to x$ is a group in which 1 is the identity and $x^{-1} = x \to 1 = x \rightsquigarrow 1$ is the inverse of $x \in G_A$. The original operations $\to$ and $\rightsquigarrow$ on G_A are retrieved from $\cdot$ by $x \to y = y \cdot x^{-1}$ and $x \rightsquigarrow y = x^{-1} \cdot y$.

Proposition

For every $a \in A$, $(a \to 1) \to 1$ is the only element $g \in G_A$ with $a \leq g$ and G_A is the set of all maximal elements of A.

Important subalgebras of pseudo-BCI-algebra $(A, \to, \rightsquigarrow, 1)$:

- Integral part of A ... $I_A = \{a \in A \mid a \leq 1\}$ - 1 is the top element of I_A, i.e. I_A is a pseudo BCK-algebra.

  $x \in I_A$ iff $((x \to 1) \to 1) \to x = x$

- Group part of A ... $G_A = \{a \to 1 \mid a \in A\}$

  $x \in G_A$ iff $((x \to 1) \to 1) = x$

Theorem

$(G_A, \cdot)$ where $x \cdot y = (x \to 1) \rightsquigarrow y = (y \rightsquigarrow 1) \to x$ is a group in which 1 is the identity and $x^{-1} = x \to 1 = x \rightsquigarrow 1$ is the inverse of $x \in G_A$. The original operations $\to$ and $\rightsquigarrow$ on G_A are retrieved from $\cdot$ by $x \to y = y \cdot x^{-1}$ and $x \rightsquigarrow y = x^{-1} \cdot y$.

Proposition

For every $a \in A$, $(a \to 1) \to 1$ is the only element $g \in G_A$ with $a \leq g$ and G_A is the set of all maximal elements of A.

Important subalgebras of pseudo-BCI-algebra $(A, \to, \rightsquigarrow, 1)$:

- Integral part of A ... $I_A = \{a \in A \mid a \leq 1\}$ - 1 is the top element of I_A, i.e. I_A is a pseudo BCK-algebra.

  $x \in I_A$ iff $((x \to 1) \to 1) \to x = x$

- Group part of A ... $G_A = \{a \to 1 \mid a \in A\}$

  $x \in G_A$ iff $((x \to 1) \to 1) = x$

Theorem

$(G_A, \cdot)$ where $x \cdot y = (x \to 1) \rightsquigarrow y = (y \rightsquigarrow 1) \to x$ is a group in which 1 is the identity and $x^{-1} = x \to 1 = x \rightsquigarrow 1$ is the inverse of $x \in G_A$. The original operations $\to$ and $\rightsquigarrow$ on G_A are retrieved from $\cdot$ by $x \to y = y \cdot x^{-1}$ and $x \rightsquigarrow y = x^{-1} \cdot y$.

Proposition

For every $a \in A$, $(a \to 1) \to 1$ is the only element $g \in G_A$ with $a \leq g$ and G_A is the set of all maximal elements of A.

Important subalgebras of pseudo-BCI-algebra $(A, \to, \rightsquigarrow, 1)$:

- Integral part of A ... $I_A = \{a \in A \mid a \leq 1\}$ - 1 is the top element of I_A, i.e. I_A is a pseudo BCK-algebra.

  $x \in I_A$ iff $((x \to 1) \to 1) \to x = x$

- Group part of A ... $G_A = \{a \to 1 \mid a \in A\}$

  $x \in G_A$ iff $((x \to 1) \to 1) = x$

Theorem

$(G_A, \cdot)$ where $x \cdot y = (x \to 1) \rightsquigarrow y = (y \rightsquigarrow 1) \to x$ is a group in which 1 is the identity and $x^{-1} = x \to 1 = x \rightsquigarrow 1$ is the inverse of $x \in G_A$. The original operations $\to$ and $\rightsquigarrow$ on G_A are retrieved from $\cdot$ by $x \to y = y \cdot x^{-1}$ and $x \rightsquigarrow y = x^{-1} \cdot y$.

Proposition

For every $a \in A$, $(a \to 1) \to 1$ is the only element $g \in G_A$ with $a \leq g$ and G_A is the set of all maximal elements of A.

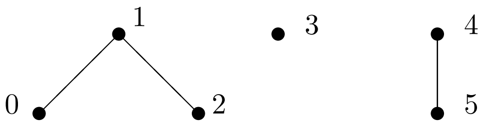
Example:

The set $A = \{0, 1, 2, 3, 4, 5\}$ equipped with the operations $\to$ and $\rightsquigarrow$ given by the following tables is a proper pseudo BCI-algebra:

$\to$	0	2	3	4	5	1
0	3	3	4	2	4	1
2	0	1	3	4	4	1
3	4	4	1	3	3	4
4	3	3	4	1	0	3
5	3	3	4	1	1	3
1	0	2	3	4	5	1

$\rightsquigarrow$	0	2	3	4	5	1
0	3	3	2	4	2	1
2	0	1	3	4	5	1
3	4	4	1	3	3	4
4	3	3	4	1	2	3
5	3	3	4	1	1	3
1	0	2	3	4	5	1

$(A, \leq)$



For the pseudo BCI-algebra we have $I_A = \{0, 1, 2\}$ and $G_A = \{1, 3, 4\}$ with the group operation $\cdot$ given by the following table:

$\cdot$	1	3	4
1	1	3	4
3	3	4	1
4	4	1	3

Group part $G_A^\star = (G_A, \rightsquigarrow, \to, 1)$:

$(G_A, \star)$, where $g \star h = h \cdot g$ for all $g, h \in G_A$, is a group isomorphic to $(G_A, \cdot)$ (isomorphism $g \mapsto g^{-1}$).

$G_A^\star = (G_A, \rightsquigarrow, \to, 1)$ ... $g \rightsquigarrow h = h \star g^{-1}$, $g \to h = g^{-1} \star h$

Remark

It is easy to show that

$$\theta = \{(x, y) \mid x \to 1 = y \to 1\}$$

is a congruence on A such that $[1]_\theta = I_A$.

Remark

The map $\alpha : a \mapsto a \to 1 = a \rightsquigarrow 1$ is a homomorphism of $(A, \to, \rightsquigarrow, 1)$ onto $(G_A, \rightsquigarrow, \to, 1)$ with kernel congruence θ, i.e. $(A, \to, \rightsquigarrow, 1)/\theta \cong (G_A, \rightsquigarrow, \to, 1)$.

The map $\beta : a \mapsto (a \to 1) \to 1$ is a homomorphism of $(A, \to, \rightsquigarrow, 1)$ onto $(G_A, \to, \rightsquigarrow, 1)$ with kernel congruence θ, i.e. $(A, \to, \rightsquigarrow, 1)/\theta \cong (G_A, \to, \rightsquigarrow, 1)$.

$x \to 1 = y \to 1$ iff $(x \to 1) \to 1 = (y \to 1) \to 1$

Remark

It is easy to show that

$$\theta = \{(x, y) \mid x \to 1 = y \to 1\}$$

is a congruence on A such that $[1]_\theta = I_A$.

Remark

The map $\alpha : a \mapsto a \to 1 = a \rightsquigarrow 1$ is a homomorphism of $(A, \to, \rightsquigarrow, 1)$ onto $(G_A, \rightsquigarrow, \to, 1)$ with kernel congruence θ, i.e. $(A, \to, \rightsquigarrow, 1)/\theta \cong (G_A, \rightsquigarrow, \to, 1)$.

The map $\beta : a \mapsto (a \to 1) \to 1$ is a homomorphism of $(A, \to, \rightsquigarrow, 1)$ onto $(G_A, \to, \rightsquigarrow, 1)$ with kernel congruence θ, i.e. $(A, \to, \rightsquigarrow, 1)/\theta \cong (G_A, \to, \rightsquigarrow, 1)$.

$x \to 1 = y \to 1$ iff $(x \to 1) \to 1 = (y \to 1) \to 1$

For any $a, b \in A$ we define
$x \circ y = (x \to 1) \rightsquigarrow y$,
$x \star y = (y \rightsquigarrow 1) \to x$.

For the operations $\circ$ and $\star$ we have

1. $1 \circ x = x$, $x \star 1 = x$,
2. $x \circ (x \to 1) = 1$, $(x \rightsquigarrow 1) \star x = 1$,
3. $x \circ (y \star z) = (x \circ y) \star z$,
4. $(x \circ y) \circ z \leq x \circ (y \circ z)$,
5. $x \star (y \star z) \leq (x \star y) \star z$.

The operation $\circ$ is associative iff $\star$ is associative.

For any $a, b \in A$ we define
$x \circ y = (x \to 1) \rightsquigarrow y$,
$x \star y = (y \rightsquigarrow 1) \to x$.

For the operations $\circ$ and $\star$ we have

1. $1 \circ x = x$, $x \star 1 = x$,
2. $x \circ (x \to 1) = 1$, $(x \rightsquigarrow 1) \star x = 1$,
3. $x \circ (y \star z) = (x \circ y) \star z$,
4. $(x \circ y) \circ z \leq x \circ (y \circ z)$,
5. $x \star (y \star z) \leq (x \star y) \star z$.

The operation $\circ$ is associative iff $\star$ is associative.

Theorem

Let $(A, \to, \rightsquigarrow, 1)$ be a pseudo BCI-algebra such that both $\circ$ and $\star$ are associative and $g \to x = g \rightsquigarrow x$ for all $g \in G_A$ and $x \in I_A$. Then $(A, \to, \rightsquigarrow, 1)$ is isomorphic to the direct product of $(I_A, \to, \rightsquigarrow, 1)$ and $(G_A, \to, \rightsquigarrow, 1)$ as well as to direct product of $(I_A, \to, \rightsquigarrow, 1)$ and $(G_A, \rightsquigarrow, \to, 1)$.

Remark

The map $\phi : I_A \times G_A^\star \to A$ defined by $\phi((x, g)) = g \to x = g \rightsquigarrow x$ is the isomorphism.

The map $\psi : I_A \times G_A \to A$ defined by $\psi((x, g)) = g^{-1} \to x = g^{-1} \rightsquigarrow x$ is the isomorphism.

Theorem

Let $(A, \rightarrow, \rightsquigarrow, 1)$ be a pseudo BCI-algebra such that both $\circ$ and $\star$ are associative and $g \rightarrow x = g \rightsquigarrow x$ for all $g \in G_A$ and $x \in I_A$. Then $(A, \rightarrow, \rightsquigarrow, 1)$ is isomorphic to the direct product of $(I_A, \rightarrow, \rightsquigarrow, 1)$ and $(G_A, \rightarrow, \rightsquigarrow, 1)$ as well as to direct product of $(I_A, \rightarrow, \rightsquigarrow, 1)$ and $(G_A, \rightsquigarrow, \rightarrow, 1)$.

Remark

The map $\phi : I_A \times G_A^{\star} \rightarrow A$ defined by
$\phi((x, g)) = g \rightarrow x = g \rightsquigarrow x$ is the isomorphism.

The map $\psi : I_A \times G_A \rightarrow A$ defined by
$\psi((x, g)) = g^{-1} \rightarrow x = g^{-1} \rightsquigarrow x$ is the isomorphism.

Prefilters and filters

A prefilter in a pseudo BCI-algebra A is $D \subseteq A$ such that

(i) $1 \in D$,

(ii) if $x \in D$ and $x \to y \in D$, then $y \in D$,

(iii) for all $x \in A$, if $x \in D$ then $x \to 1 \in D$.

Lemma

Let $(A, \to, \rightsquigarrow, 1)$ be a pseudo BCI-algebra.

1. Any prefiltr is an order-filter, i.e., $x \in D$ and $x \leq y$ imply $y \in D$.
2. Any prefilter is a subalgebra of $(A, \to, \rightsquigarrow, 1)$.
3. $D \subseteq A$ is a prefilter if and only if $1 \in D$ and

(ii') for all $x, y \in A$, if $x \rightsquigarrow y \in D$ and $x \in D$ then $y \in D$.

(iii') for all $x \in A$, if $x \in D$ then $x \rightsquigarrow 1 \in D$.

Prefilters and filters

A prefilter in a pseudo BCI-algebra A is $D \subseteq A$ such that

(i) $1 \in D$,

(ii) if $x \in D$ and $x \to y \in D$, then $y \in D$,

(iii) for all $x \in A$, if $x \in D$ then $x \to 1 \in D$.

Lemma

Let $(A, \to, \rightsquigarrow, 1)$ be a pseudo BCI-algebra.

1. Any prefiltr is an order-filter, i.e., $x \in D$ and $x \leq y$ imply $y \in D$.
2. Any prefilter is a subalgebra of $(A, \to, \rightsquigarrow, 1)$.
3. $D \subseteq A$ is a prefilter if and only if $1 \in D$ and

(ii') for all $x, y \in A$, if $x \rightsquigarrow y \in D$ and $x \in D$ then $y \in D$.

(iii') for all $x \in A$, if $x \in D$ then $x \rightsquigarrow 1 \in D$.

A prefilter D is a filter if, for all $x, y \in A$,

$$x \to y \in D \quad \text{iff} \quad x \rightsquigarrow y \in D.$$

Denote $\mathscr{F}(A)$ the set of all filters on A.

I_A is always a filter of A.

Let $\mathscr{K}$ be a class of algebras of type F, $A \in \mathscr{K}$ and $\theta \in \mathrm{Con}(A)$.

We say that θ is a *relative congruence* (or *$\mathscr{K}$-congruence*) on A if $A/\theta \in \mathscr{K}$.

Denote $\mathrm{Con}_{\mathscr{K}}(A)$ the set of all relative congruences on A.

The filters correspond to the relative congruences: for every filter D,

$$\theta_D = \{(x, y) \mid x \to y \in D \text{ and } y \to x \in D\}$$

is the only relative congruence such that $[1]_{\theta_D} = D$.

A prefilter D is a filter if, for all $x, y \in A$,

$$x \to y \in D \quad \text{iff} \quad x \rightsquigarrow y \in D.$$

Denote $\mathscr{F}(A)$ the set of all filters on A.
I_A is always a filter of A.
Let $\mathscr{K}$ be a class of algebras of type F, $A \in \mathscr{K}$ and $\theta \in \mathrm{Con}(A)$.
We say that θ is a *relative congruence* (or *$\mathscr{K}$-congruence*) on A if $A/\theta \in \mathscr{K}$.
Denote $\mathrm{Con}_{\mathscr{K}}(A)$ the set of all relative congruences on A.

The filters correspond to the relative congruences: for every filter D,

$$\theta_D = \{(x, y) \mid x \to y \in D \text{ and } y \to x \in D\}$$

is the only relative congruence such that $[1]_{\theta_D} = D$.

A prefilter D is a filter if, for all $x, y \in A$,

$$x \to y \in D \quad \text{iff} \quad x \rightsquigarrow y \in D.$$

Denote $\mathscr{F}(A)$ the set of all filters on A.
I_A is always a filter of A.
Let $\mathscr{K}$ be a class of algebras of type F, $A \in \mathscr{K}$ and $\theta \in \mathrm{Con}(A)$.
We say that θ is a *relative congruence* (or *$\mathscr{K}$-congruence*) on A if $A/\theta \in \mathscr{K}$.
Denote $\mathrm{Con}_{\mathscr{K}}(A)$ the set of all relative congruences on A.

The filters correspond to the relative congruences: for every filter D,

$$\theta_D = \{(x, y) \mid x \to y \in D \text{ and } y \to x \in D\}$$

is the only relative congruence such that $[1]_{\theta_D} = D$.

A prefilter D is a filter if, for all $x, y \in A$,

$$x \to y \in D \quad \text{iff} \quad x \rightsquigarrow y \in D.$$

Denote $\mathscr{F}(A)$ the set of all filters on A.
I_A is always a filter of A.
Let $\mathscr{K}$ be a class of algebras of type F, $A \in \mathscr{K}$ and $\theta \in \mathrm{Con}(A)$.
We say that θ is a *relative congruence* (or *$\mathscr{K}$-congruence*) on A if $A/\theta \in \mathscr{K}$.
Denote $\mathrm{Con}_{\mathscr{K}}(A)$ the set of all relative congruences on A.

The filters correspond to the relative congruences: for every filter D,

$$\theta_D = \{(x, y) \mid x \to y \in D \text{ and } y \to x \in D\}$$

is the only relative congruence such that $[1]_{\theta_D} = D$.

Theorem

Let $(A, \to, \rightsquigarrow, 1)$ be a pseudo BCI-algebra. Then $(A, \to, \rightsquigarrow, 1)$ is isomorphic to the direct product of $(I_A, \to, \rightsquigarrow, 1)$ and $(G_A, \to, \rightsquigarrow, 1)$ as well as to direct product of $(I_A, \to, \rightsquigarrow, 1)$ and $(G_A, \rightsquigarrow, \to, 1)$ iff G_A is a filter on A.

Theorem

Let $(G, \cdot)$ be a group with the identity e and define $x \to y = y \cdot x^{-1}$ and $x \rightsquigarrow y = x^{-1} \cdot y$. Then $(G, \to, \rightsquigarrow, e)$ is a trivially ordered pseudo BCI- algebra where $\emptyset \neq H \subseteq G$ is a prefilter iff it is a subgroup of $(G, \cdot)$ and H is a filter iff it is a normal subgroup of $(G, \cdot)$.

Corollary

The lattice of prefilters need not be modular.
The lattice of filters need not be distributive.

Theorem

Let $(A, \to, \rightsquigarrow, 1)$ be a pseudo BCI-algebra. Then $(A, \to, \rightsquigarrow, 1)$ is isomorphic to the direct product of $(I_A, \to, \rightsquigarrow, 1)$ and $(G_A, \to, \rightsquigarrow, 1)$ as well as to direct product of $(I_A, \to, \rightsquigarrow, 1)$ and $(G_A, \rightsquigarrow, \to, 1)$ iff G_A is a filter on A.

Theorem

Let $(G, \cdot)$ be a group with the identity e and define $x \to y = y \cdot x^{-1}$ and $x \rightsquigarrow y = x^{-1} \cdot y$. Then $(G, \to, \rightsquigarrow, e)$ is a trivially ordered pseudo BCI- algebra where $\emptyset \neq H \subseteq G$ is a prefilter iff it is a subgroup of $(G, \cdot)$ and H is a filter iff it is a normal subgroup of $(G, \cdot)$.

Corollary

The lattice of prefilters need not be modular.
The lattice of filters need not be distributive.

Properties of quasivariety of pseudo BCI-algebras

1) Quasivariety of pseudo BCI-algebras is relatively regular in 1

A quasi-variety $\mathscr{K}$ with a constant term 1 is said to be *relatively regular in 1*, if $[1]_\theta = [1]_\phi$ implies $\theta = \phi$ for all $\theta, \phi \in \mathrm{Con}_{\mathscr{K}}(A)$.

It is known that a quasi-variety $\mathscr{K}$ is relatively regular in 1 iff there exist the terms $d_1(x, y), \ldots, d_n(x, y)$ in $\mathscr{K}$ such that $d_1(x, y) = 1, \ldots, d_n(x, y) = 1$ implies $x = y$.

Obviously, for the quasi-variety of all pseudo-BCI-algebras we can take $n = 2$, $d_1(x, y) = x \to y$ and $d_2(x, y) = y \to x$.

2) Quasivariety of pseudo BCI-algebras is relatively ideal determined

Let $\mathscr{K}$ be a class of algebras of type F with a constant 1. A term $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ of type F is called an *ideal term* in $y_1, \ldots, y_n$ if $\mathscr{K} \models t(x_1, \ldots, x_m, 1, \ldots, 1) = 1$.
A non-empty subset I of A is said to be *closed under the ideal term* $t(x_1, \ldots, x_m, y_1, \ldots, y_n)$ in $y_1, \ldots, y_n$ if $t(a_1, \ldots, a_m, b_1, \ldots, b_n) \in I$ whenever $b_1, \ldots, b_n \in I$.
We say that $\emptyset \neq I \subseteq A$ is an *ideal* in A if it is closed under all ideal terms.
The class $\mathscr{K}$ is called *relatively ideal determined* if for each $A \in \mathscr{K}$, every ideal in A is the kernel of a unique relative congruence on A.

Denote $\mathscr{I}(A)$ the set of all ideals on A.

Theorem

Let A be a pseudo BCI-algebra and $I \subseteq A$ with $1 \in I$. The following statements are equivalent:

(i) I is a filter of A.

(ii) $I = [1]_\theta$ for some $\theta \in \mathrm{Con}_{\mathscr{K}}(A)$.

(iii) I is an ideal of A.

(iv) I is closed with respect to the ideal terms

$$t_1(x_1, x_2, y_1, y_2) = (((y_1 \to (y_2 \to x_1)) \to x_1) \rightsquigarrow x_2) \rightsquigarrow x_2$$

$$t_2(y) = y \to 1$$

(v) I is closed with respect to the ideal terms

$$w_1(x, y_1, y_2) = (y_1 \to (y_2 \to x)) \to x$$

$$w_2(x, y) = (y \rightsquigarrow x) \rightsquigarrow x$$

$$t_2(y) = y \to 1$$

Corollary

$\mathrm{Con}_{\mathscr{K}}(A) \cong \mathscr{I}(A) = \mathscr{F}(A)$.

Let $\mathscr{K}$ be a relatively regular in 1 quasivarity in which there is a one-one correspondence between ideals and relative congruences, that is, for every algebra $A \in \mathscr{K}$, the map $\theta \mapsto [1]_\theta$ is an isomorphism of $\mathrm{Con}_{\mathscr{K}}(A)$ onto $\mathscr{I}(A)$. Then the following lemma holds:

Lemma

Let $\alpha, \beta \in \mathrm{Con}_{\mathscr{K}}(A)$. Then
$[1]_\alpha \vee [1]_\beta = \{a \in A \mid (a, b) \in \alpha$ for some $b \in [1]_\beta\} =$
$= \{a \in A \mid (a, b) \in \beta$ for some $b \in [1]_\alpha\}$, i.e. $a \in [1]_\alpha \vee [1]_\beta$ iff $(a, 1) \in \alpha \circ \beta$ iff $(a, 1) \in \beta \circ \alpha$.

Corollary

The lattices $\mathrm{Con}_{\mathscr{K}}(A) \cong \mathscr{I}(A) = \mathscr{F}(A)$ are modular.

W. J. Blok, J. G. Raftery: Ideals in quasivarieties of algebras, Models, Algebras and Proofs, Lecture Notes in Pure and Applied Mathematics, Vol. 203, Marcel Dekker, New York, 1999, 167-186.

W. J. Blok, J. G. Raftery: On the quasivariety of BCK-algebras and its subvarieties, Alg. Univer. Vol. 33, No. 1, 68-90.

W. A. Dudek, Y. B. Jun: Pseudo BCI-algebras. East Asian Mathematical Journal, 2008, 24(2), 187-190.

G. Georgescu, A. Iorgulescu: Pseudo-BCK algebras: An extension of BCK algebras, Combinatorics, Computability and Logic, Proc. DMTCS'01, Springer, London, 2001, 97-114.

V. A. Gorbunov: Algebraic Theory of Quasivarieties, Siberian school of algebra and logic, Plenum Publishing Corporation, New York, 1998.

R. Halaš: Annihilators in BCK-algebras, Czech. Math. J. 53 (2003), 1001-1007.

R. Halaš, J. Kühr: Deductive systems and annihilators of pseudo BCK-algebras, Italian J. of Pure and Appl. Math. 25 (2009),

A. Iorgulescu: On pseudo-BCK algebras and porims, Sci. Math. Japon. 10 (2004), 293-305.

K. Iseki, On BCI-algebras, Math. Seminar Notes, 8 (1980), 125130.

K. Iseki and S. Tanaka, An introduction to the theory of BCK-algebras, Math. Japonica, 23 (1978), 126.

Y. B. Jun, H. S. Kim, J. Neggers: On pseudo-BCI ideals of pseudo-BCI algebras, Matematicki Vestnik, 58 (2006), 39-46.

J. Kühr: Pseudo BCK-algebras and residuated lattices, Contr. Gen. Alg. 16 (2005), 139-144.

J. G. Raftery, C. J. van Alten: Residuation in commutative ordered monoids with minimal zero, Reports on Mathematical Logic, 34 (2000), 23-57.

A. Wroński: BCK-algebras do not form a variety, Math. Jap. 28 (1983), 211-213.

THANK YOU
FOR YOUR ATTENTION!