Prove that $(x^2 + y^2 + z^2)^2 \geq 3x^3y + 3y^3z + 3z^3x$

Joint work with Mátyás Domokos

Eötvös Loránd University, Budapest

March, 2009

Ross Willard, early '90-s of last century

computer algebra class

- $p_1(x_1, x_2, \dots x_n)$ and $p_2(x_1, x_2, \dots, x_n)$
- Are p_1 and p_2 equal at every substitution?

After class he started to think about what he did. Then looked up the literature

Theorem

Hunts, **Stearnes** *Journal of Symbolic Logic, (1990)*
Let R be a commutative ring. The above problem can be decided in polynomal time, if R is nilpotent, and coNP complete otherwise.

Heard on the street

A highschool math problem

$x^2 + xy + y^2 = 9;$
$y^2 + yz + z^2 = 16;$
$x^2 + xz + z^2 = 25$
$x, y, z > 0$

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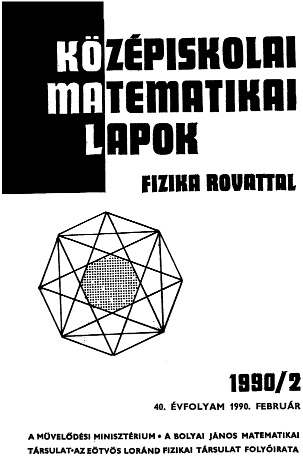
Other version

$x^2 + xy + y^2 = 2;$
$y^2 + yz + z^2 = 3;$
$x^2 + xz + z^2 = 5$


KÖZÉPISKOLAI MATEMATIKAI LAPOK

FIZIKA ROVATTAL

1990/2

40. ÉVFOLYAM 1990. FEBRUÁR

A MŰVELŐDÉSI MINISZTÉRIUM • A BOLYAI JÁNOS MATEMATIKAI TÁRSULAT•AZ EÖTVÖS LORÁND FIZIKAI TÁRSULAT FOLYÓIRATA

Highschool Math Journal

FORUM: Interesting Problems

The problem

Prove that

$(x^2 + y^2 + z^2)^2 \geq 3x^3y + 3y^3z + 3z^3x$

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My solution:

$a^3b = \frac{a}{b}a^2b^2$ so apply Jensen's inequality:
Let f be a convex function. Then

$$f(\sum \lambda_i x_i) \leq \sum \lambda_i f(x_i))$$

for $f(x) = \sqrt{x}$

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To obtain

$$(x^2 + y^2 + z^2)^2 \leq (\frac{x}{y} + \frac{y}{z} + \frac{z}{x})(3x^3y + 3y^3z + 3z^3x)$$

Official solution

Observe that

$$\frac{1}{2}\big((x^2+2yz-y^2-zx-xy)^2+$$
$$(y^2+2zx-z^2-xy-yz)^2+$$
$$+(z^2+2xy-x^2-yz-zx)^2\big)=$$

$$=x^4+y^4+z^4+2x^2y^2+2x^2z^2+2y^2z^2-3x^3y-3y^3z-3z^3x$$

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!!&!!!??!?!

Official solution

Equality holds iff

1. $x = y = z$,
2. $x : y : z = \sin^2(\frac{4\pi}{7}) : \sin^2(\frac{2\pi}{7}) : \sin^2(\frac{\pi}{7})$.
3. $y : z : x = \sin^2(\frac{4\pi}{7}) : \sin^2(\frac{2\pi}{7}) : \sin^2(\frac{\pi}{7})$.
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!!&!!!??!?!

No standard techinque works

First approach

Bilinear form over $Z[x^2, y^2, z^2, xy, xz, yz]$

$$B(x,y,z) = x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2 - 3x^3y - 3y^3z - 3z^3x$$

$$\begin{pmatrix} 1 & a & b & s & r & t \\ a & 1 & c & t & s & r \\ b & c & 1 & r & t & s \\ s & t & r & d & f & g \\ r & s & t & f & d & e \\ t & r & s & g & e & d \end{pmatrix}$$

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$x^2y^2 = x^2 \cdot y^2 = xy \cdot xy : 2a + d = 2$
$x^2yz = x^2 \cdot yz = xy \cdot xz : t + s = 0$
Lucky:
$xy^3 = x^2 \cdot xy : s = 0$

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The matrix

$$A = \begin{pmatrix} 1 & a & b & -1,5 & 0 & t \\ a & 1 & c & 0 & r & -1,5 \\ b & c & 1 & s & -1,5 & 0 \\ -1,5 & 0 & s & 2-2a & -t & -r \\ 0 & r & -1,5 & -t & 2-2b & -s \\ t & -1,5 & 0 & -r & -s & 2-2c \end{pmatrix}$$

Find a, b, c, r, s, t such that A is a positive semidefinite. Then use Gram-Schmidt.

Our solution

$a = b = c - \frac{1}{2}$, $r = s = t - \frac{3}{2}$

$rank(A) = 2$

$$(x^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2 - \frac{3}{2}xy + \frac{3}{2}yz)^2 + \frac{3}{4}(y^2 - z^2 - yz + 2xz - xy)^2 =$$
$$= x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2 - 3x^3y - 3y^3z - 3z^3x,$$

We beat the original official solution

Bugging questions

Questions

Which parameters correspond to the original solution?
Any other representations?

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Answers: E. Bogár, BSC Thesis

The same

Not in less than a sum of 4 squares

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Techniques

Determinant Resultant

Hilbert's 17th

Theorem (Hilbert)

Let $F_k(\bar{x}(n))$ an n-variable form of degree k such that

$$F_k(\bar{x}(n)) \geq 0 \quad \text{for every } \bar{x} \in \mathbb{R}^n.$$

Then F can always be written as a sum of squares
if and only if $n = 2$ or $k = 4$ and $n = 3$.

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Is it true that F can always be written as a sum of squares?

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Then F can always be written as a sum of squares of polynomials if and only if $n = 2$ or $k = 4$ and $n = 3$.

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Is it true that F can always be written as a sum of squares of rational functions?

Solution

Artin 1927

Yes

Solution

Artin 1927

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Hilbert

For $n = 3$ and $k = 4$ three squares are always enough.

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- Hilbert: non-constructive geometric arguments (Motzkin 1965)

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- Later: The decision problem of being positive is decidable

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Techniques

- Hilbert: non-constructive geometric arguments (Motzkin 1965)
- Artin: Developed the Artin-Schreier theory
- Later: The decision problem of being positive is decidable
- There are non-effective algorithms

Solution

For $n = 3$ and $k = 4$ (V. Powers & al) 2006

$$\begin{pmatrix} 1 & a & b & s & r & t \\ a & 1 & c & t & s & r \\ b & c & 1 & r & t & s \\ s & t & r & d & f & g \\ r & s & t & f & d & e \\ t & r & s & g & e & d \end{pmatrix}$$

There are (maximum) 8 (eight) proper choice of the variables.

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Techniques

Picquard-group of smooth curves
Weil-divisors

A phone call

Mátyás Domokos

This concerns the theory of invariants.

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Technique

Finding homogeneous components of forms vanishing on $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(1,0,0)$,

A phone call

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This concerns the theory of invariants.

Technique

Finding homogeneous components of forms vanishing on $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(1,0,0)$,

Result

If $F_4(x,y,z)$ has 4 roots, then it is a sum of two squares. (Converse true over $\mathbb{C}$ by Bezout's Theorem)

Phone call

For cyclically symmetric forms:

Z_3 invariance (over $\mathbb{R}$) implies that the form is the sum of two squares.

Phone call

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Techniques

Irreducible representations over $\mathbb{R}$.
Finding the (two) invariant subspaces

Results:

For cyclically symmetric forms:

Finding the (unique) form

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Finding the (unique) form

1. in terms of its roots:
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Finding the (unique) form

1. in terms of its roots:
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2. in terms of the coefficients of $p(x) = (x-l)(x-n)(x-m)$

Results:

For cyclically symmetric forms:

Finding the (unique) form

1. in terms of its roots:
   $(0,0,0)$, (l,m,n), (m,n,l), (n,l,m),
2. in terms of the coefficients of $p(x) = (x-l)(x-n)(x-m)$
3. as a sum of two squares.

An example:

Let $F = (x^4 + y^4 + z^4) + d(x^2y^2 + y^2z^2 + z^2x^2) + 2\alpha(x^3y + y^3z + z^3x) + 2\beta(y^3x + z^3y + x^3z) + \lambda(x^2yz + xy^2z + xyz^2)$, where $\alpha \neq \beta$
Then d and λ are determined and

$$F = (x^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2 + \alpha xy + \beta xz - (\alpha + \beta)yz)^2 + \frac{3}{4}(y^2 - z^2 + \frac{2}{3}(2\beta + \alpha)xy + \frac{2}{3}(\alpha - \beta)yz - \frac{2}{3}(\beta + 2\alpha)xz)^2$$

Observe that $(1, 1, 1)$ is always a root.

A new highschool problem:

Prove that

$$(x^4 + y^4 + z^4) + \sqrt{3}(x^3y + y^3z + z^3x) \geq (\sqrt{3} + 1)xyz(x + y + z)$$

A new highschool problem:

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$$(x^4 + y^4 + z^4) + \sqrt{3}(x^3y + y^3z + z^3x) \geq (\sqrt{3} + 1)xyz(x + y + z)$$

Solution

Observe that

$$F = \left(x^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2 + \sqrt{3}xy + (\sqrt{3})yz\right)^2 + \frac{3}{4}\left(y^2 - z^2 + \frac{2\sqrt{3}}{3}xy + \frac{2\sqrt{3}}{3}yz - \frac{4\sqrt{3}}{3}xz\right)^2$$

Groups

	ID-CHECK	ext. ID-CHECK
nilpotent	P	P
solvable	? ?	coNP-complete, f
non-solvable	coNP-complete	coNP-complete
S_3	P	coNP-complete, $[,]$?
A_4	P	coNP-complete, $[,]$
S_4	?	coNP-complete, $[,]$

Official solution

Equality holds iff

1. $x = y = z$,
2. $x : y : z = a : b : c$ for one of the two cyclic orders of the roots a, b, c of
   $x^3 + (6 - \sqrt{3})x^2 - 4\sqrt{3}x - 1$

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Two beers

If you find a nice form of λa,λb,λc, where a, b, c are the roots of $x^3 + (6 - \sqrt{3})x^2 - 4\sqrt{3}x - 1$