Functional reducts of the countable atomless Boolean algebra

Kende Kalina
joint work with Bertalan Bodor, Michael Pinsker and Csaba Szabó

ELTE, Hungary

07th September 2014

Different signatures for the same structure

Boolean algebra

A Boolean algebra has five operations: $\wedge, \vee, \neg, \bot, \top$

Different signatures for the same structure

Boolean algebra

A Boolean algebra has five operations: $\wedge, \vee, \neg, \bot, \top$

Boolean ring

A Boolean ring has four operations: $\cdot, +, 0, 1$

Different signatures for the same structure

Boolean algebra

A Boolean algebra has five operations: $\wedge, \vee, \neg, \bot, \top$

Boolean ring

A Boolean ring has four operations: $\cdot, +, 0, 1$

Equivalence

$a + b = (\neg a \wedge b) \vee (a \wedge \neg b)$

Weaker sets of operations

If we omit $\cdot, \neg$ and 1 we get:

Weaker sets of operations

If we omit $\cdot, \neg$ and 1 we get:

Vector space

A structure with two operations: $+, 0$

Weaker sets of operations

If we omit $\cdot, \neg$ and 1 we get:

Vector space

A structure with two operations: $+, 0$

Weaker sets of operations

If we omit $\cdot, \neg$ and 1 we get:

Vector space

A structure with two operations: $+, 0$

Not equivalent!

Weaker sets of operations

If we omit $\cdot, \neg$ and 1 we get:

Vector space

A structure with two operations: $+, 0$

Not equivalent! Why?

Weaker sets of operations

If we omit $\cdot, \neg$ and 1 we get:

Vector space

A structure with two operations: $+, 0$

Not equivalent! Why?

Automorphism groups

Weaker sets of operations

If we omit $\cdot, \neg$ and 1 we get:

Vector space

A structure with two operations: $+, 0$

Not equivalent! Why?

Automorphism groups

1. $\mathrm{Aut}\,(BA) \subsetneq \mathrm{Aut}\,(Vec)$

Weaker sets of operations

If we omit $\cdot, \neg$ and 1 we get:

Vector space

A structure with two operations: $+, 0$

Not equivalent! Why?

Automorphism groups

1. $\text{Aut}\,(BA) \subsetneq \text{Aut}\,(Vec)$
2. There is a permutation preserving $+$ and 0 but not preserving some of $\cdot, \neg, 1$

Weaker sets of operations

If we omit $\cdot, \neg$ and 1 we get:

Vector space

A structure with two operations: $+, 0$

Not equivalent! Why?

Automorphism groups

1. $\mathrm{Aut}\,(BA) \subsetneq \mathrm{Aut}\,(Vec)$
2. There is a permutation preserving $+$ and 0 but not preserving some of $\cdot, \neg, 1$
3. $+$ and 0 can not define $\cdot, \neg, 1$

Weaker sets of operations

If we omit everything but 0 we get:

Weaker sets of operations

If we omit everything but 0 we get:

Pointed set

A structure with one constant: 0

Weaker sets of operations

If we omit everything but 0 we get:

Pointed set

A structure with one constant: 0

Weaker sets of operations

If we omit everything but 0 we get:

Pointed set

A structure with one constant: 0

Not equivalent with the Boolean algebra because

Weaker sets of operations

If we omit everything but 0 we get:

Pointed set

A structure with one constant: 0

Not equivalent with the Boolean algebra because

Automorphism groups

Weaker sets of operations

If we omit everything but 0 we get:

Pointed set

A structure with one constant: 0

Not equivalent with the Boolean algebra because

Automorphism groups

1. $\mathrm{Aut}\,(BA) \subsetneq \mathrm{Sym}(BA)_0$

Weaker sets of operations

If we omit everything but 0 we get:

Pointed set

A structure with one constant: 0

Not equivalent with the Boolean algebra because

Automorphism groups

1. $\mathrm{Aut}\,(BA) \subsetneq \mathrm{Sym}(BA)_0$
2. There is a permutation preserving 0 but not preserving some of $\cdot, \neg, 1$

Functional reducts

Definition

$\mathcal{B}$ is a functional reduct of $\mathcal{A}$ if

Functional reducts

Definition

$\mathcal{B}$ is a functional reduct of $\mathcal{A}$ if

Functional reducts

Definition

$\mathcal{B}$ is a functional reduct of $\mathcal{A}$ if

- The clone of term functions of $\mathcal{A}$ contains the clone of term functions of $\mathcal{B}$

Functional reducts

Definition

$\mathcal{B}$ is a functional reduct of $\mathcal{A}$ if

- The clone of term functions of $\mathcal{A}$ contains the clone of term functions of $\mathcal{B}$

Functional reducts

Definition

$\mathcal{B}$ is a functional reduct of $\mathcal{A}$ if

- The clone of term functions of $\mathcal{A}$ contains the clone of term functions of $\mathcal{B}$
- If f is a term function over $\mathcal{A}$, and f can be defined over $\mathcal{B}$, then f must be a term function over $\mathcal{B}$

Functional reducts

Definition

$\mathcal{B}$ is a functional reduct of $\mathcal{A}$ if

- The clone of term functions of $\mathcal{A}$ contains the clone of term functions of $\mathcal{B}$
- If f is a term function over $\mathcal{A}$, and f can be defined over $\mathcal{B}$, then f must be a term function over $\mathcal{B}$

Functional reducts

Definition

$\mathcal{B}$ is a functional reduct of $\mathcal{A}$ if

- The clone of term functions of $\mathcal{A}$ contains the clone of term functions of $\mathcal{B}$
- If f is a term function over $\mathcal{A}$, and f can be defined over $\mathcal{B}$, then f must be a term function over $\mathcal{B}$

Arbitrary first order definitions are allowed!

Functional reducts

Definition

$\mathcal{B}$ is a functional reduct of $\mathcal{A}$ if

- The clone of term functions of $\mathcal{A}$ contains the clone of term functions of $\mathcal{B}$
- If f is a term function over $\mathcal{A}$, and f can be defined over $\mathcal{B}$, then f must be a term function over $\mathcal{B}$

Arbitrary first order definitions are allowed!

Like $(a \vee b = c) \Leftrightarrow (\forall d)((ad = a \text{ and } bd = d) \rightarrow (cd = c))$

Functional reducts

Definition

$\mathcal{B}$ is a functional reduct of $\mathcal{A}$ if

- The clone of term functions of $\mathcal{A}$ contains the clone of term functions of $\mathcal{B}$
- If f is a term function over $\mathcal{A}$, and f can be defined over $\mathcal{B}$, then f must be a term function over $\mathcal{B}$

Arbitrary first order definitions are allowed!

Like $(a \vee b = c) \Leftrightarrow (\forall d)((ad = a \text{ and } bd = d) \to (cd = c))$

Functional reducts

Definition

$\mathcal{B}$ is a functional reduct of $\mathcal{A}$ if

- The clone of term functions of $\mathcal{A}$ contains the clone of term functions of $\mathcal{B}$
- If f is a term function over $\mathcal{A}$, and f can be defined over $\mathcal{B}$, then f must be a term function over $\mathcal{B}$

Arbitrary first order definitions are allowed!

Like $(a \vee b = c) \Leftrightarrow (\forall d)((ad = a \text{ and } bd = d) \to (cd = c))$

Functional reducts of the countable atomless Boolean algebra

Functional reducts of the countable atomless Boolean algebra

Theorem (B., K., Sz.)

Up to (first-order) equivalence the countable atomless Boolean algebra has 13 fuctional reducts

Functional reducts of the countable atomless Boolean algebra

Theorem (B., K., Sz.)

Up to (first-order) equivalence the countable atomless Boolean algebra has 13 fuctional reducts

Functional reducts of the countable atomless Boolean algebra

Theorem (B., K., Sz.)

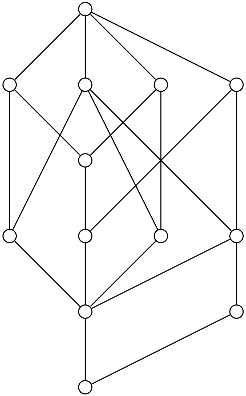
Up to (first-order) equivalence the countable atomless Boolean algebra has 13 fuctional reducts
(For each stucture we give a basis for the clone of operations)

Functional reducts of the countable atomless Boolean algebra

Theorem (B., K., Sz.)

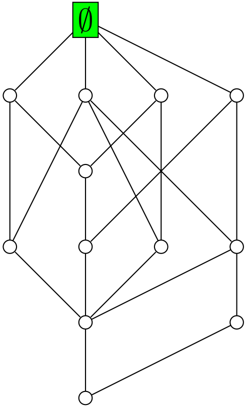
Up to (first-order) equivalence the countable atomless Boolean algebra has 13 fuctional reducts
(For each stucture we give a basis for the clone of operations)
Same classification holds for all Boolean algebra with enough elements

Functional reducts of the countable atomless Boolean algebra



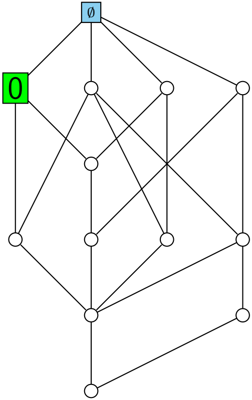
Functional reducts of the countable atomless Boolean algebra

1. Set: no operations



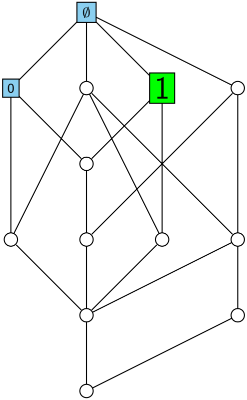
Functional reducts of the countable atomless Boolean algebra

1. Set: no operations
2. One constant: 0



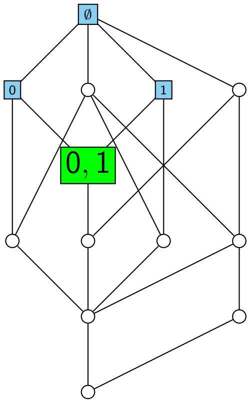
Functional reducts of the countable atomless Boolean algebra

1. Set: no operations
2. One constant: 0
3. One constant: 1



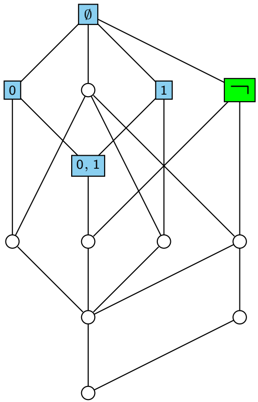
Functional reducts of the countable atomless Boolean algebra

1. Set: no operations
2. One constant: 0
3. One constant: 1
4. Two constants: 0, 1



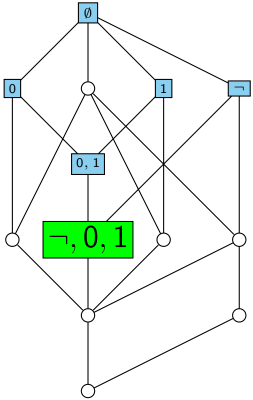
Functional reducts of the countable atomless Boolean algebra

1. Set: no operations
2. One constant: 0
3. One constant: 1
4. Two constants: $0, 1$
5. Complement: $\neg x = x + 1$



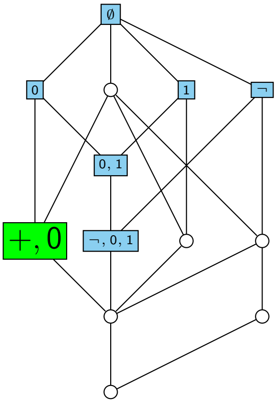
Functional reducts of the countable atomless Boolean algebra

1. Set: no operations
2. One constant: 0
3. One constant: 1
4. Two constants: $0, 1$
5. Complement: $\neg x = x + 1$
6. Complement and two constants: $\neg, 0, 1$



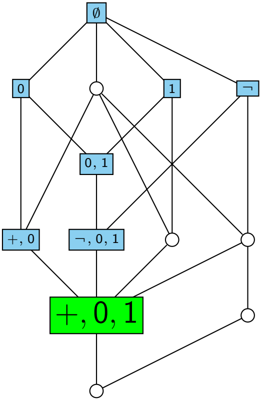
Functional reducts of the countable atomless Boolean algebra

1. Set: no operations
2. One constant: 0
3. One constant: 1
4. Two constants: $0, 1$
5. Complement: $\neg x = x + 1$
6. Complement and two constants: $\neg, 0, 1$
7. Vector space: $+, 0$



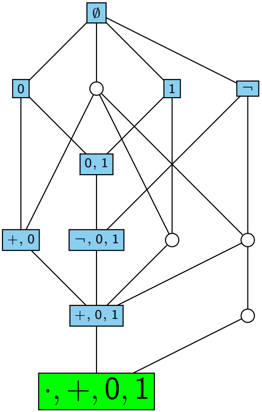
Functional reducts of the countable atomless Boolean algebra

1. Set: no operations
2. One constant: 0
3. One constant: 1
4. Two constants: $0, 1$
5. Complement: $\neg x = x + 1$
6. Complement and two constants: $\neg, 0, 1$
7. Vector space: $+, 0$
8. Vector space with a constant: $+, 0, 1$



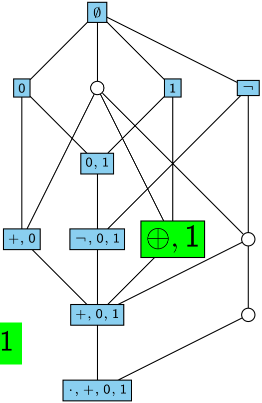
Functional reducts of the countable atomless Boolean algebra

1. Set: no operations
2. One constant: 0
3. One constant: 1
4. Two constants: $0, 1$
5. Complement: $\neg x = x + 1$
6. Complement and two constants: $\neg, 0, 1$
7. Vector space: $+, 0$
8. Vector space with a constant: $+, 0, 1$
9. Boolean algebra: $\cdot, +, 0, 1$



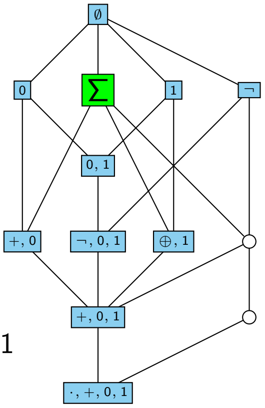
Functional reducts of the countable atomless Boolean algebra

1. Set: no operations
2. One constant: 0
3. One constant: 1
4. Two constants: $0, 1$
5. Complement: $\neg x = x + 1$
6. Complement and two constants: $\neg, 0, 1$
7. Vector space: $+, 0$
8. Vector space with a constant: $+, 0, 1$
9. Boolean algebra: $\cdot, +, 0, 1$
10. Vector space No.2: $a \oplus b = a + b + 1, 1$



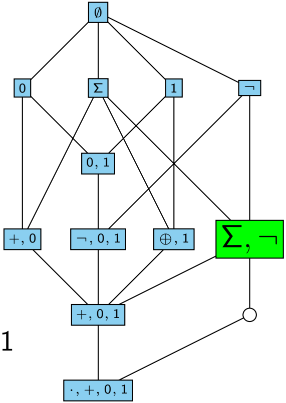
Functional reducts of the countable atomless Boolean algebra

1. Set: no operations
2. One constant: 0
3. One constant: 1
4. Two constants: $0, 1$
5. Complement: $\neg x = x + 1$
6. Complement and two constants: $\neg, 0, 1$
7. Vector space: $+, 0$
8. Vector space with a constant: $+, 0, 1$
9. Boolean algebra: $\cdot, +, 0, 1$
10. Vector space No.2: $a \oplus b = a + b + 1, 1$
11. Affine space: $\Sigma(a, b, c) = a + b + c$



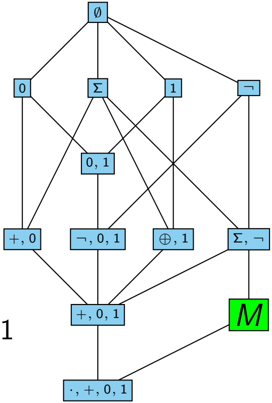
Functional reducts of the countable atomless Boolean algebra

1. Set: no operations
2. One constant: 0
3. One constant: 1
4. Two constants: $0, 1$
5. Complement: $\neg x = x + 1$
6. Complement and two constants: $\neg, 0, 1$
7. Vector space: $+, 0$
8. Vector space with a constant: $+, 0, 1$
9. Boolean algebra: $\cdot, +, 0, 1$
10. Vector space No.2: $a \oplus b = a + b + 1, 1$
11. Affine space: $\Sigma(a, b, c) = a + b + c$
12. Affine space with complement: $\Sigma, \neg$



Functional reducts of the countable atomless Boolean algebra

1. Set: no operations
2. One constant: 0
3. One constant: 1
4. Two constants: $0, 1$
5. Complement: $\neg x = x + 1$
6. Complement and two constants: $\neg, 0, 1$
7. Vector space: $+, 0$
8. Vector space with a constant: $+, 0, 1$
9. Boolean algebra: $\cdot, +, 0, 1$
10. Vector space No.2: $a \oplus b = a + b + 1, 1$
11. Affine space: $\Sigma(a, b, c) = a + b + c$
12. Affine space with complement: $\Sigma, \neg$
13. Structure with median: $M(a, b, c) = a \cdot b + b \cdot c + c \cdot a$



Vector space No.2

Operations

$a \oplus b = a + b + 1,1$

Vector space No.2

Operations

$a \oplus b = a + b + 1, 1$

"Upside down"

$\phi(x) = \neg x$ is an isomorphism with the vector space $+, 0$

Affine space

Operations

$\Sigma(a, b, c) = a + b + c$

Affine space

Operations

$\Sigma(a, b, c) = a + b + c$

Geometric meaning

The relation $R(a, b, c, d) \Leftrightarrow d = \Sigma(a, b, c) \Leftrightarrow a + b + c + d = 0$ is symmetrical

Affine space

Operations

$\Sigma(a, b, c) = a + b + c$

Geometric meaning

The relation $R(a, b, c, d) \Leftrightarrow d = \Sigma(a, b, c) \Leftrightarrow a + b + c + d = 0$ is symmetrical

Affine space

Operations

$\Sigma(a, b, c) = a + b + c$

Geometric meaning

The relation $R(a, b, c, d) \Leftrightarrow d = \Sigma(a, b, c) \Leftrightarrow a + b + c + d = 0$ is symmetrical

R means that $\{a, b, c, d\}$ is a 2-dimensional affine subspace

Automorphisms

$\text{Aut}\,(\mathit{Aff}) = \text{Aut}\,(\mathit{Vec}) \ltimes T$

Structure with median

Operations

$M(a, b, c) = a \cdot b + b \cdot c + c \cdot a$

Structure with median

Operations

$M(a, b, c) = a \cdot b + b \cdot c + c \cdot a$

Structure with median

Operations

$M(a, b, c) = a \cdot b + b \cdot c + c \cdot a$
$= (a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$

Structure with median

Operations

$M(a, b, c) = a \cdot b + b \cdot c + c \cdot a$
$= (a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$

Automorphisms

Structure with median

Operations

$M(a, b, c) = a \cdot b + b \cdot c + c \cdot a$
$= (a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$

Automorphisms

Structure with median

Operations

$M(a,b,c) = a \cdot b + b \cdot c + c \cdot a$
$=(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$

Automorphisms

Let T be the set of all translations $t_a(x) = x + a$

Structure with median

Operations

$M(a, b, c) = a \cdot b + b \cdot c + c \cdot a$
$=(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$

Automorphisms

Let T be the set of all translations $t_a(x) = x + a$
A permutation preserves M iff it is a composition of a (BA) automorphism and a translation

Structure with median

Operations

$M(a, b, c) = a \cdot b + b \cdot c + c \cdot a$
$=(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$

Automorphisms

Let T be the set of all translations $t_a(x) = x + a$
A permutation preserves M iff it is a composition of a (BA) automorphism and a translation
$\mathrm{Aut}\,(M) = \mathrm{Aut}\,(BA) \ltimes T$

Structure with median

Operations

$M(a,b,c) = a \cdot b + b \cdot c + c \cdot a$
$= (a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$

Automorphisms

Let T be the set of all translations $t_a(x) = x + a$
A permutation preserves M iff it is a composition of a (BA) automorphism and a translation
$\mathrm{Aut}\,(M) = \mathrm{Aut}\,(BA) \ltimes T$
Reminder: $\mathrm{Aut}\,(Aff) = \mathrm{Aut}\,(Vec) \ltimes T$

Structure with median

Operations

$M(a, b, c) = a \cdot b + b \cdot c + c \cdot a$
$=(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$

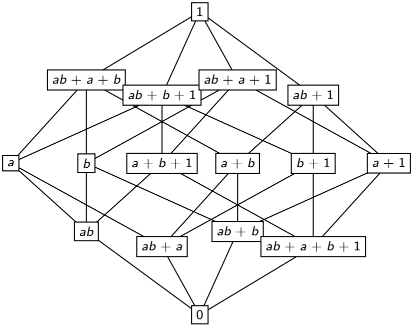
Automorphisms

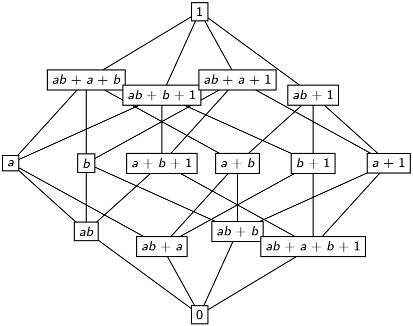
Let T be the set of all translations $t_a(x) = x + a$
A permutation preserves M iff it is a composition of a (BA) automorphism and a translation
$\mathrm{Aut}\,(M) = \mathrm{Aut}\,(BA) \ltimes T$
Reminder: $\mathrm{Aut}\,(Aff) = \mathrm{Aut}\,(Vec) \ltimes T$

Intervals

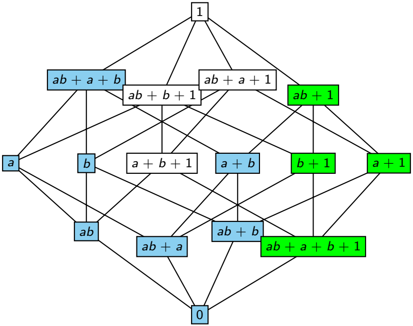
A permutation preserves M iff it preserves intervals

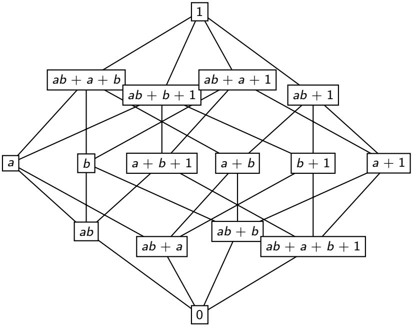
The action of a translation



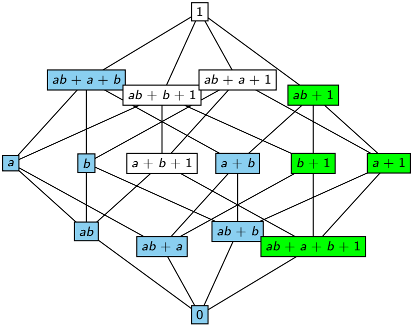


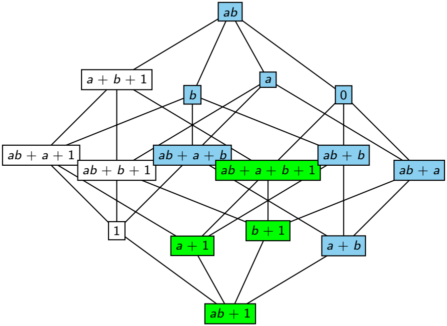
The action of a translation





Translation by $ab + 1$





Amongst all reducts

Amongst all reducts

Reduct

Relations which are first-order definable

Amongst all reducts

Reduct

Relations which are first-order definable

Amongst all reducts

Reduct

Relations which are first-order definable

For ω-categorical structures:

Bijection with the closed groups containing the automorphisms

Amongst all reducts

Reduct

Relations which are first-order definable

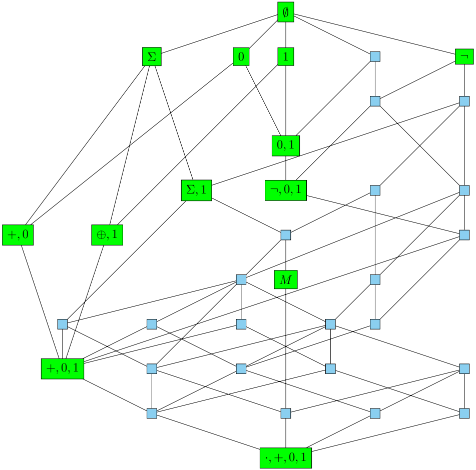
For ω-categorical structures:

Bijection with the closed groups containing the automorphisms

Problem

Characterisation of functional reducts amongst all reducts

Known reducts of the countable atomless Boolean algebra



Amongst all reducts

Amongst all reducts

Observations

Amongst all reducts

Observations

Amongst all reducts

Observations

Not all reducts are functional

Amongst all reducts

Observations

Not all reducts are functional
But they form a sublattice!

Amongst all reducts

Observations

Not all reducts are functional

But they form a sublattice!

Question

Always a sublattice?

Counterexample

Counterexample

The structure

Counterexample

The structure

Counterexample

The structure

ϕ, ψ: two unary operations on a countable infinite set such that

Counterexample

The structure

ϕ, ψ: two unary operations on a countable infinite set such that

- For all x elements $\phi^3(x) = \psi^3(x) = x$

Counterexample

The structure

ϕ, ψ: two unary operations on a countable infinite set such that

- For all x elements $\phi^3(x) = \psi^3(x) = x$
- For all x elements $\phi^2(x) \neq x$ and $\psi^2(x) \neq x$

Counterexample

The structure

ϕ, ψ: two unary operations on a countable infinite set such that

- For all x elements $\phi^3(x) = \psi^3(x) = x$
- For all x elements $\phi^2(x) \neq x$ and $\psi^2(x) \neq x$
- There are infinitely many x such that $\phi^2(x) = \psi^2(x)$

Counterexample

The structure

ϕ, ψ: two unary operations on a countable infinite set such that

- For all x elements $\phi^3(x) = \psi^3(x) = x$
- For all x elements $\phi^2(x) \neq x$ and $\psi^2(x) \neq x$
- There are infinitely many x such that $\phi^2(x) = \psi^2(x)$
- There are infinitely many x such that $\phi^2(x) \neq \psi^2(x)$

Counterexample

The structure

ϕ, ψ: two unary operations on a countable infinite set such that

- For all x elements $\phi^3(x) = \psi^3(x) = x$
- For all x elements $\phi^2(x) \neq x$ and $\psi^2(x) \neq x$
- There are infinitely many x such that $\phi^2(x) = \psi^2(x)$
- There are infinitely many x such that $\phi^2(x) \neq \psi^2(x)$

Counterexample

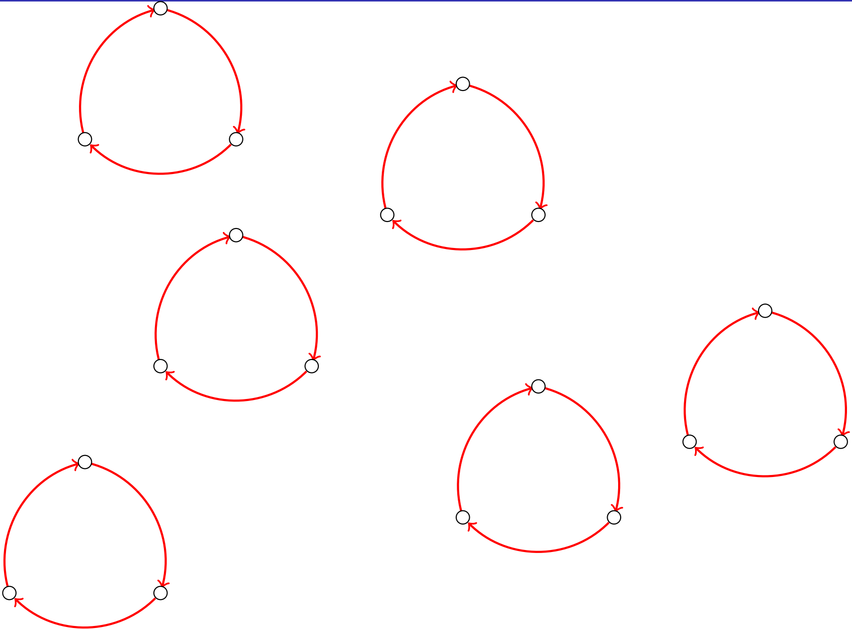
The structure

ϕ, ψ: two unary operations on a countable infinite set such that

- For all x elements $\phi^3(x) = \psi^3(x) = x$
- For all x elements $\phi^2(x) \neq x$ and $\psi^2(x) \neq x$
- There are infinitely many x such that $\phi^2(x) = \psi^2(x)$
- There are infinitely many x such that $\phi^2(x) \neq \psi^2(x)$

The functional reducts generated by ϕ and ψ have different joins as a reduct and as a functional reduct.

Counterexample



Counterexample