Coupled right orthosemirings induced by orthomodular lattices

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It is well-known that MV-algebras play a crucial role in the algebraic axiomatization of so-called many-valued Lukasiewicz logics. This is the reason why MV-algebras were intensively studied in the last decades. The connection between MV-algebras and certain semirings was developed by A. Di Nola and B. Gerla ([6] and [7]), see also [1] for the sheaf representation. They recognized that to every MV-algebra there can be assigned a certain triplet which consists of two semirings, one lattice ordered and the other one dually lattice ordered, and an involutory isomorphism between them. Although this construction which gives a full characterization is interesting, it is more remarkable that, with slight modifications, the mentioned construction can be used also for basic algebras and commutative basic algebras (see [4] and [5]).

Orthomodular lattices play a similar role for the logic of quantum mechanics as MV-algebras do for Lukasiewicz's many-valued logic. In fact, orthomodular lattices originated in the 1930's where G. Birkhoff and J. von Neumann used them in order to describe quantum events. They considered certain operators in Hilbert spaces and the lattice of closed subspaces of a Hilbert space. Hence, the natural question arises if a similar construction to that in [6] and [7] using semilattice-like structures can be used in order to find a characterization of orthomodular lattices. Surprisingly, this is not only possible but we can use the machinary derived in [5] with very small modifications. The resulting structure which fully characterizes orthomodular lattices is called a coupled right orthosemiring which is analogous to the coupled semirings used in [1], [6] and [7]. Hence we will show that our way of representing algebras which are axiomatizations of certain propositional logics is universal in the sense that it does not depend on the fact that the underlying logic is a many-valued logic or the logic of quantum mechanics. Both constructions are similar to each other and differ only in the underlying axioms.

Definition 1

An *orthomodular lattice* (see e.g. [2] and [9]) is an algebra $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ satisfying the following axioms:

(i) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.

(ii) $x \leq y$ implies $x' \geq y'$.

(iii) $(x')' = x$

(iv) $x \leq y$ implies $y = x \vee (y \wedge x')$.

Remark 2

Condition (iv) is called the *orthomodular law*. From this law it follows that $x \vee x' = x \vee (1 \wedge x') = 1$, i. e. x' is a lattice-theoretical complement of x.

In every orthomodular lattice there can be defined the so-called commutation relation:

Definition 3

Let $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ be an orthomodular lattice and $a, b \in L$. The elements a and b are said to commute with each other, in signs $a \mathrm{C} b$, if $a = (a \wedge b) \vee (a \wedge b')$.

This commutation relation has the following properties:

Lemma 4

(*cf.* [9]) *Let* $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ *be an orthomodular lattice and* $a, b, c \in L$*. Then the following hold:*

(i) *If* $a \mathrm{C} b$ *then* $b \mathrm{C} a$*.*

(ii) *If* $a \leq b$ *then* $a \mathrm{C} b$*.*

(iii) *If* $a \mathrm{C} b$ *then* $a \mathrm{C} b'$*.*

(iv) *If two of the three elements* a*,* b *and* c *commute with the third one then* $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ *and* $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$*.*

The concept of a right near semiring was introduced by the authors in [5]. For the reader's convenience we repeat this definition.

Definition 5

A *right near semiring* is an algebra $\mathcal{R} = (R, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ satisfying the following axioms:

(i) $(R, +, 0)$ is a commutative monoid.

(ii) $(R, \cdot, 1)$ is a groupoid with neutral element.

(iii) $(x + y)z = xz + yz$ for all $x, y, z \in R$.

(iv) $x0 = 0x = 0$ for all $x \in R$.

Let $\mathcal{R} = (R, +, \cdot, 0, 1)$ be a right near semiring. $\mathcal{R}$ is called

- $\vee$-*semilattice ordered* if there exists a join-semilattice operation $\vee$ on R such that $x + y = x \vee y$ and $xy \leq y$ for all $x, y \in R$ with respect to the induced order.
- $\wedge$-*semilattice ordered* if there exists a meet-semilattice operation $\wedge$ on R such that $x + y = x \wedge y$ and $xy \geq y$ for all $x, y \in R$ with respect to the induced order.
- a *near semiring* if $\cdot$ is distributive with respect to $+$.
- *commutative* if $\cdot$ is commutative.

Remark 6

Condition (iii) is called the *right distributive law*. Of course, every commutative right near semiring is a commutative near semiring (in the sense of [4]).

Next we define coupled right orthosemirings.

Definition 7

A *coupled right orthosemiring* is an ordered triple

$$((R, \vee, \cdot, 0, 1), (R, \wedge, *, 1, 0), \alpha)$$

satisfying the following conditions:

(R1) $(R, \vee, \wedge)$ is a lattice.

(R2) $(R, \vee, \cdot, 0, 1)$ is a $\vee$-semilattice ordered right near semiring.

(R3) $(R, \wedge, *, 1, 0)$ is a $\wedge$-semilattice ordered right near semiring.

(R4) α is an involutory isomorphism between $(R, \vee, \cdot, 0, 1)$ and $(R, \wedge, *, 1, 0)$.

(R5) $(x \wedge \alpha(y)) \vee y = x * y$ for all $x, y \in R$

(R6) $y * (x \wedge y) = y$ for all $x, y \in R$

Now we can assign to every orthomodular lattice a coupled right orthosemiring in some natural way.

Theorem 8

Let $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ *be an orthomodular lattice and define two binary operations* $\oplus$ *and* $\odot$ *on* L *by*

$$x \oplus y := (x \wedge y') \vee y$$

and

$$x \odot y := (x \vee y') \wedge y$$

for all $x, y \in L$. *Then*

$$\mathbf{N}(\mathcal{L}) := ((L, \vee, \odot, 0, 1), (L, \wedge, \oplus, 1, 0), ')$$

is a coupled right orthosemiring.

Proof.

Let $a, b, c \in L$. (R1) and (R4) are evident.

(R2) Obviously, $(L, \vee, 0)$ is a commutative monoid. Moreover we have $a \odot 1 = (a \vee 1') \wedge 1 = a \vee 0 = a$ and
$1 \odot a = (1 \vee a') \wedge a = 1 \wedge a = a$.

Hence $(L, \odot, 1)$ is a groupoid with neutral element. Moreover, using Lemma 4 we obtain
$(a \vee b) \odot c = ((a \vee b) \vee c') \wedge c = (a \vee b \vee c') \wedge c$
$= ((a \vee c') \vee (b \vee c')) \wedge c = ((a \vee c') \wedge c) \vee ((b \vee c') \wedge c)$
$= (a \odot c) \vee (b \odot c)$.

Finally, $a \odot 0 = (a \vee 0') \wedge 0 = 0$ and
$0 \odot a = (0 \vee a') \wedge a = a' \wedge a = 0$.

This shows that $(L, \vee, \odot, 0, 1)$ is a right near semiring.
Because of $a \odot b = (a \vee b') \wedge b \leq b$, this right near semiring is $\vee$-semilattice ordered.

Continuation of the proof.

(R3) Obviously, $(L, \wedge, 1)$ is a commutative monoid. Moreover we have $a \oplus 0 = (a \wedge 0') \vee 0 = a \wedge 1 = a$ and
$0 \oplus a = (0 \wedge a') \vee a = 0 \vee a = a$.

Hence $(L, \oplus, 0)$ is a groupoid with neutral element. Moreover, using Lemma 4 we obtain
$(a \wedge b) \oplus c = ((a \wedge b) \wedge c') \vee c = (a \wedge b \wedge c') \vee c$
$= ((a \wedge c') \wedge (b \wedge c')) \vee c = ((a \wedge c') \vee c) \wedge ((b \wedge c') \vee c)$
$= (a \oplus c) \wedge (b \oplus c)$.

Finally, $a \oplus 1 = (a \wedge 1') \vee 1 = 1$ and
$1 \oplus a = (1 \wedge a') \vee a = a' \vee a = 1$.
This shows that $(L, \wedge, \oplus, 1, 0)$ is a right near semiring.
Because of $a \oplus b = (a \wedge b') \vee b \geq b$, this right near semiring is $\wedge$-semilattice ordered.

Continuation of the proof.

(R5) follows from the definition of $\oplus$.

(R6) According to orthomodularity we have

$$b \oplus (a \wedge b) = (b \wedge (a \wedge b)') \vee (a \wedge b) = b.$$

□

Next we are going to prove the converse of Theorem 8, i.e. we will show that to each coupled right orthosemiring we can assign an orthomodular lattice in a natural way.

Theorem 9

Let $\mathcal{N} = ((R, \vee, \cdot, 0, 1), (R, \wedge, *, 1, 0), \alpha)$ *be a coupled right orthosemiring. Then* $\mathbf{L}(\mathcal{N}) := (R, \vee, \wedge, \alpha, 0, 1)$ *is an orthomodular lattice.*

Proof.

Let $a, b \in R$. According to (R1), $(R, \vee, \wedge)$ is a lattice. Because of (R2) and (R3), $(R, \vee, 0)$ and $(R, \wedge, 1)$ are monoids and hence $(R, \vee, \wedge, 0, 1)$ is a bounded lattice. According to (R4), α is an antitone involution on this bounded lattice. If $a \leq b$ then

$$a \vee (b \wedge \alpha(a)) = (b \wedge \alpha(a)) \vee a = b * a = b * (a \wedge b) = b$$

according to (R5) and (R6). □

Finally, we can show that the just introduced correspondence between orthomodular lattices and coupled right orthosemirings is one-to-one.

Theorem 10

Let $\mathcal{L} = (L, \vee, \wedge, ', 0, 1)$ *be an orthomodular lattice. Then* $\mathbf{L}(\mathbf{N}(\mathcal{L})) = \mathcal{L}$.

Let $\mathcal{N} = ((R, \vee, \cdot, 0, 1), (R, \wedge, *, 1, 0), \alpha)$ *be a coupled right orthosemiring. Then* $\mathbf{N}(\mathbf{L}(\mathcal{N})) = \mathcal{N}$.

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