Reducts of the infinite dimensional vector space

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ELTE, Hungary

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Two reduct are called interdefinable iff they are reducts of one another.

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Theorem ("folklore")

For "random" structures this is a bijection.

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Ramsey theory: Bodirsky, Pinsker, etc.

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Hence: Old techniques do not work.
Two cases: $\mathbb{F}_2$; $\mathbb{F}_p$, for $p \geq 3$ (prime).

Theorem (B.,K.,Sz.)

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1. the vector space $\mathcal{V}$ itself

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Reducts of the vector space

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1. the vector space $\mathcal{V}$ itself
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3. the countably infinite dimensional affine space

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1. $\mathrm{Aut}(\mathcal{V})$, the automorphism group of $\mathcal{V}$
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3. $\mathrm{Aff}(\mathcal{V}) = \mathrm{Aut}(\mathcal{V}) \ltimes \mathrm{Tr}$, the group of affine transformations on $\mathcal{V}$

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Model theoretical formulation

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3. the countably infinite dimensional affine space
4. the countably infinite set with a constant 0

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4. $\mathrm{Sym}(\mathcal{V})_0$, the stabilizer of 0 in $\mathrm{Sym}(\mathcal{V})$

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If $\mathrm{Aut}(\mathcal{V}) \leq G \leq \mathrm{Sym}(\mathcal{V})$ *a closed group which does not fix 0, then either* $G = \mathrm{Aff}(\mathcal{V})$ *or* $G = \mathrm{Sym}(\mathcal{V})$.

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In examples: $p = 7$.

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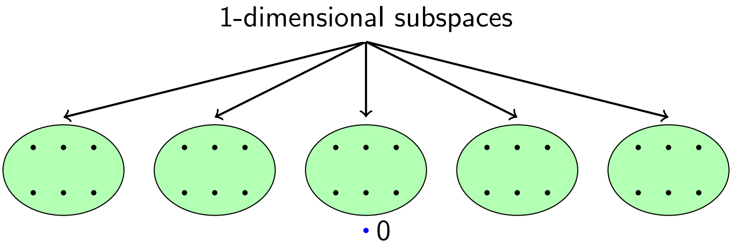

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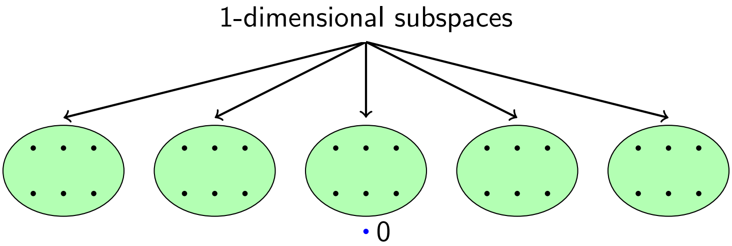
G_1 acts on the projective space $\mathcal{P} = \mathcal{V}/(x \sim \lambda x)$.



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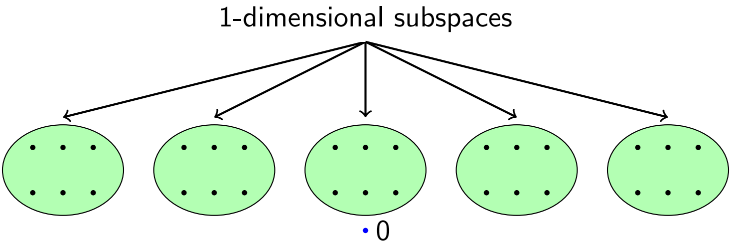
Action on $\mathcal{P}$: $\mathrm{Sym}(\mathcal{P})$.



Example 2

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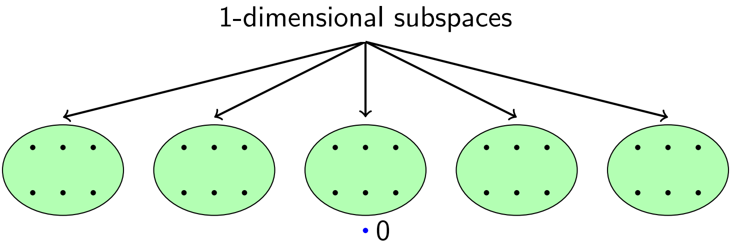

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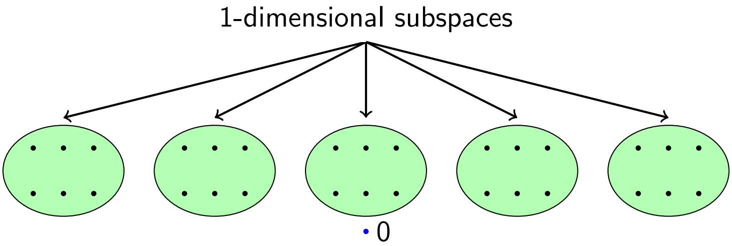

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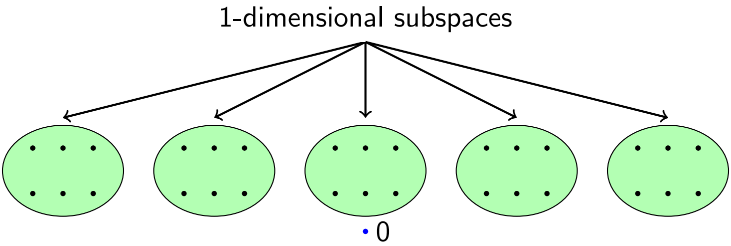
Action on $\mathcal{P}$: $\mathrm{PGL}(\mathcal{P})$ (same as for $\mathrm{Aut}(\mathcal{V})$).



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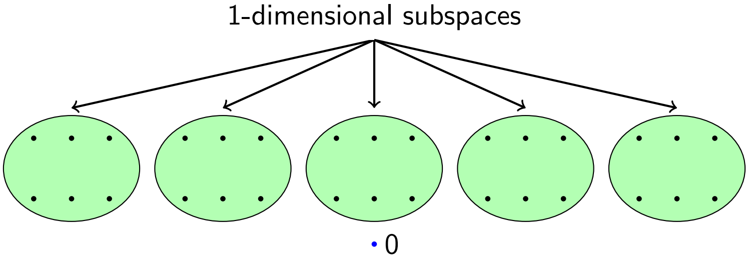

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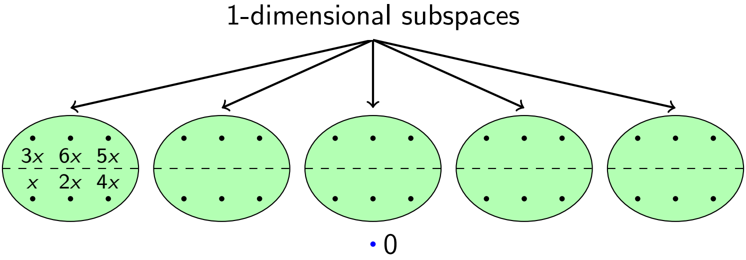

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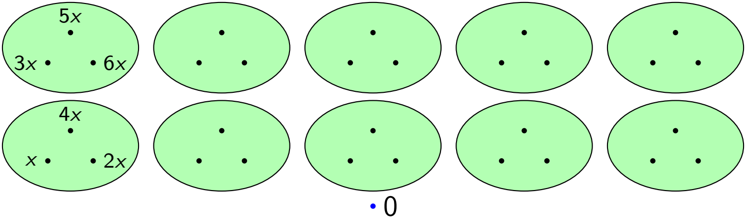
The R_3-equivalence classes



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Action on the R_3-equivalence classes: symmetric.

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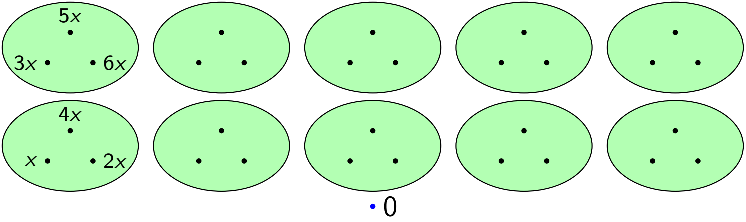

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Action within the R_3-equivalence classes: S_3.

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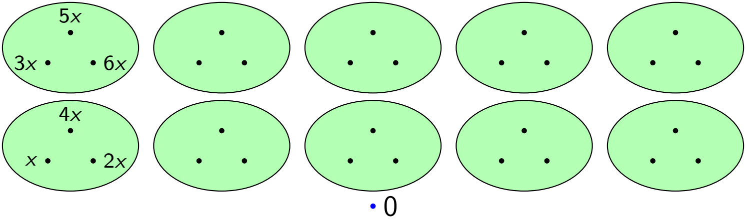

Example 4

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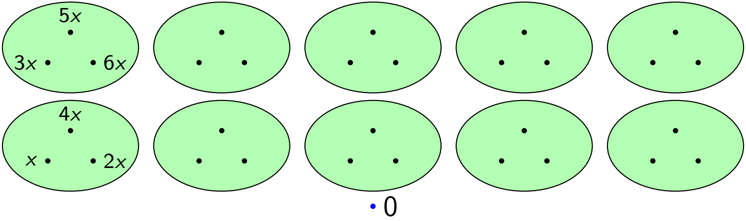

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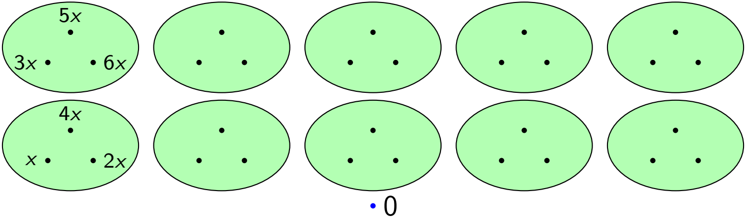

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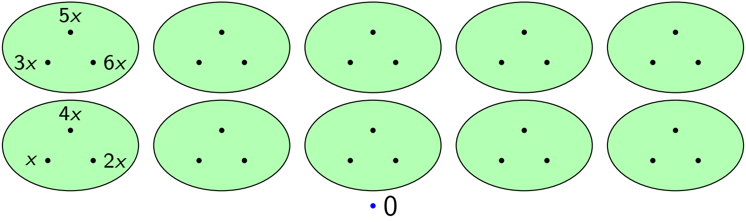

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Action within the R_3-equivalence classes: cyclic (Z_3).

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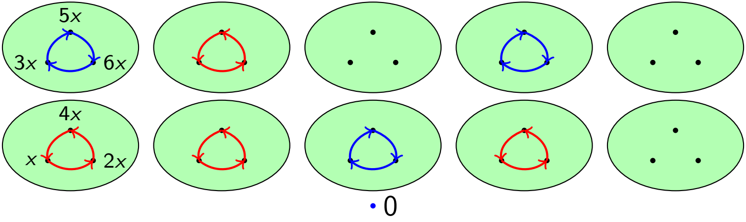

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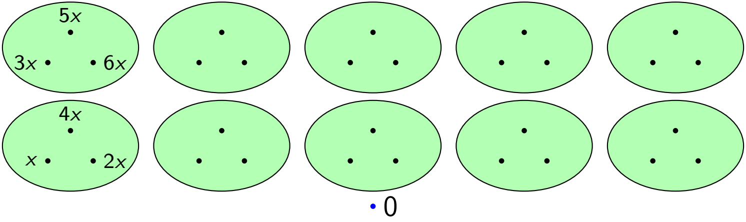

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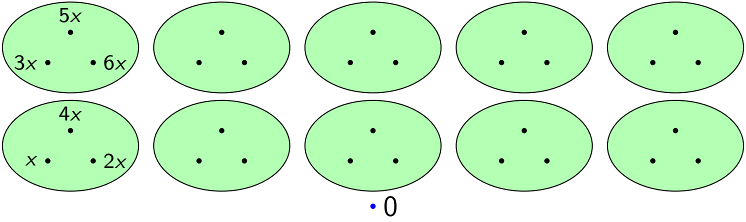

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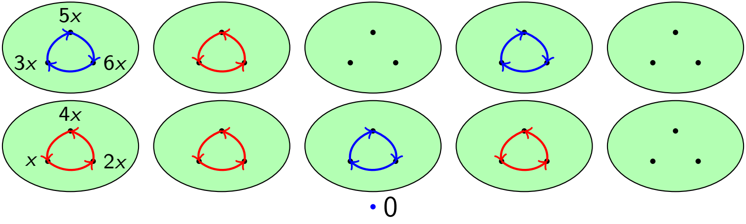

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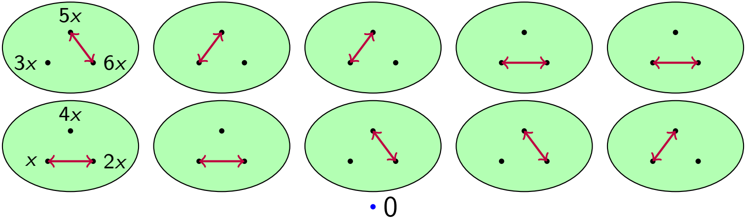

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Moreover, in the latter case there exist groups $N \triangleleft H < S_k$ such that $g \in G$ iff the action of g within each "$x \sim \lambda^k x$" equivalence class is an element of H, and they are all equal modulo N.