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Centraliser clones on finite sets & the Burris-Willard-conjecture[1]

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Outline

1. A Galois theory based on commutation
2. The Burris–Willard conjecture
3. Why cdeg $(k) \leq 1 + k^t$, $t = k^4 - k^3 + k^2$?

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Notation

Finitary operations

- For $k \in \mathbb{N}_+$ a func $f\colon A^k \longrightarrow A$ is a k-ary operation on A
- $\mathrm{O}_A^{(k)} := A^{A^k}$ set of k-ary operations on A
- $\mathrm{O}_A := \bigcup_{k\in\mathbb{N}_+} \mathrm{O}_A^{(k)}$ set of all finitary operations on A

Basis of the Galois connection

Commutation

For $n, m \in \mathbb{N}_+$, $f \in \mathrm{O}_A^{(n)}$, $g \in \mathrm{O}_A^{(m)}$, we define

$$f \perp g \;:\Longleftrightarrow$$

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$$f \perp g :\Longleftrightarrow \forall X = (x_{i,j})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \in A^{m \times n}:$$

$$\begin{matrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{matrix}$$

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$$\begin{array}{ccc} g & & g \\ \frown & & \frown \\ x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \cdots & x_{m,n} \\ \smile & & \smile \\ = & & = \\ c_1 & \cdots & c_n \end{array}$$

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$$\begin{array}{cccccc} & \overbrace{}^{g} & & \overbrace{}^{g} & & \\ f(& x_{1,1} & \cdots & x_{1,n} &) = & r_1 \\ & \vdots & \ddots & \vdots & & \vdots \\ f(& x_{m,1} & \cdots & x_{m,n} &) = & r_m \\ & \underbrace{}_{=} & & \underbrace{}_{=} & & \\ & c_1 & \cdots & c_n & & \end{array}$$

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 & = & & = & & = \\
f(& c_1 & \cdots & c_n) & = & y
\end{array}$$

Remarks on the definition

Remark 1

Defining for $m, n \in \mathbb{N}_+$, $f \in \mathrm{O}_A^{(n)}$ and any $X = (x_{i,j})_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} \in A^{m\times n}$

$$f(X) := \begin{pmatrix} f\,(X\,(1,\cdot)) \\ \vdots \\ f\,(X\,(m,\cdot)) \end{pmatrix} = \begin{pmatrix} f\,(x_{1,1},\dots,x_{1,n}) \\ \vdots \\ f\,(x_{m,1},\dots,x_{m,n}) \end{pmatrix}$$

(f row-wise)

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we have for $f \in \mathrm{O}_A^{(n)}$, $g \in \mathrm{O}_A^{(m)}$:

$$f \perp g \iff \forall Y = (y_{i,j})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \in A^{m \times n} \colon g\left(f\left(Y\right)\right) = f\left(g\left(Y^T\right)\right).$$

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Corollary (since $\left(X^T\right)^T = X$)

For $f, g \in \mathrm{O}_A$: $f \perp g \iff g \perp f$.

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Remark 2

For $n, m \in \mathbb{N}_+$, $f \in \mathrm{O}_A^{(n)}$, $g \in \mathrm{O}_A^{(m)}$:

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For $n, m \in \mathbb{N}_+$, $f \in O_A^{(n)}$, $g \in O_A^{(m)}$:
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For $n, m \in \mathbb{N}_+$, $f \in O_A^{(n)}$, $g \in O_A^{(m)}$:

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Remark 2

For $n, m \in \mathbb{N}_+$, $f \in O_A^{(n)}$, $g \in O_A^{(m)}$:

$$f \perp g \iff \langle A; f, g\rangle \models \quad f(g(X(\cdot,1)),\dots,g(X(\cdot,n))) \approx g(f(X(1,\cdot)),\dots,f(X(1,\cdot)))$$

Characterisations of commutation

Lemma (preservation of graphs)

For $f, g \in \mathrm{O}_A$ *we have*

$$f \perp g \iff f \rhd g^{\bullet} = \{(\mathbf{x}, g(\mathbf{x})) \mid \mathbf{x} \in A^{\mathrm{ar}(g)}\}.$$

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Lemma (homomorphisms of direct powers)

For $n, m \in \mathbb{N}_+$, $f \in \mathrm{O}_A^{(n)}$, $g \in \mathrm{O}_A^{(m)}$, *we have*

$$f \perp g \iff f \colon \langle A; g \rangle^n \longrightarrow \langle A; g \rangle \text{ is a homomorphism.}$$

Galois correspondence induced by commutation

Definition

For $F \subseteq O_A$ we put

$$F^* := \{g \in O_A \mid \forall f \in F : g \perp f\} \qquad \text{(centraliser of } F\text{)}$$

$$F^{**} \qquad \text{(bicentraliser of } F\text{, bicentrical closure of } F\text{)}$$

$$\mathcal{C}_A := \{F^* \mid F \subseteq O_A\} \qquad \text{(lattice of centraliser clones)}$$

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Alternatively

For $F \subseteq \mathrm{O}_A$ we have $\qquad (F^\bullet := \{f^\bullet \mid f \in F\})$

$$F^* = \mathrm{Pol}_A F^\bullet \qquad \text{(centralisers are clones!)}$$

$$= \{g \in \mathrm{O}_A \mid g^\bullet \in \mathrm{Inv}_A F\}$$

Cardinality of the lattice of centralisers

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Theorem (Burris/Willard, 1987)

$|\mathcal{C}_A| < \aleph_0$.

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Theorem (Burris/Willard, 1987)

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State of the art

That is **all** we know!!!

Outline

Consequences of finiteness of $\mathcal{C}_A$, $|A| = k < \aleph_0$

Lemma

For $F \in \mathcal{C}_A$ there is $n(F) \in \mathbb{N}_+$ with $F = F^{(m)**}$ for $m \geq n(F)$.

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Since $\mathcal{C}_A$ is finite. . . put $\operatorname{cdeg}(k) := \max\{n(F) \mid F \in \mathcal{C}_A\}$

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Corollary

$\exists N = \operatorname{cdeg}(k)\ \forall F \in \mathcal{C}_A\colon F = F^{(N)**}$.

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Question

What is $\mathrm{cdeg}\colon \mathbb{N}_+ \longrightarrow \mathbb{N}_+$?

Consequences. . .

. . . or why knowledge of cdeg(k) is useful.

Lemma

For $n \in \mathbb{N}_+$ tfae:

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For $n \in \mathbb{N}_+$ tfae:

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4. $\forall F \subseteq \mathrm{O}_A\colon \quad F^{*(n)^*} = F^{**}.$
5. $\forall F \in \mathcal{C}_A\colon \quad F^{*(n)^*} = F.$
6. $\forall F, G \subseteq \mathrm{O}_A\colon \quad F^{*(n)} = G^{*(n)} \iff F^* = G^*.$

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6. $\forall F, G \subseteq \mathrm{O}_A: \quad F^{*(n)} = G^{*(n)} \iff F^* = G^*.$
7. $\forall F, G \in \mathcal{C}_A: \quad F^{(n)} = G^{(n)} \iff F = G.$

The Burris–Willard conjecture

Theorem (Burris/Willard, 1987)

$\forall k \in \mathbb{N}_+ \colon \operatorname{cdeg}(k) \leq 1 + k^t$ *where* $t := k^4 - k^3 + k^2$.

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A little more seems to be known...

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"*By slightly different methods we can show that any primitive positive clone on a k-element set is generated by its members of arity at most* k^k,..."

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"*By slightly different methods we can show that any primitive positive clone on a k-element set is generated [using **] by its members of arity at most* k^k*,...*"

Burris–Willard conjecture

"*...but this seems to us to be far from the best possible result (which we conjecture to be k for* $k \geq 3$*).*"

Why $k \geq 3$ in the conjecture?

Example

For $k = 2$, let $g(x, y, z) := x \ominus y \oplus z$ (minority function).

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- $F := \langle g \rangle_{O_A}$ minimal clone,
- g minimal function (generator of minimum arity).
- $\implies F^{(2)} = J_A^{(2)}$
- $\implies {F^{(k)}}^{**} = {F^{(2)}}^{**} = J_A \subsetneq F = F^{**}$.

Outline

Lemmas about quaternary rel's

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Lemma

Let $\varrho_0 := \{(x, y, u, v) \in A^4 \mid x = y \implies u = v\}$.
$|\varrho_0| = t = k^4 - k^3 + k^2$.

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Let $\varrho_0 := \{(x, y, u, v) \in A^4 \mid x = y \implies u = v\}$.
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Corollary

Let $\varrho \subseteq A^4$ *satisfy* $x = y \implies u = v$ *for* $(x, y, u, v) \in \varrho$.
Then $|\varrho| \leq t$.

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Corollary

Let $\varrho \subseteq A^4$ *satisfy* $x = y \implies u = v$ *for* $(x, y, u, v) \in \varrho$.
Then $|\varrho| \leq t$.

Proof.

$\varrho \subseteq \varrho_0 \implies |\varrho| \leq |\varrho_0| = t.$ □

Another theorem of Burris and Willard. . .

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Theorem (Burris/Willard, 1987, in the proof of Thm. 2)

Let $F, G \subseteq \mathrm{O}_A$ be clones.
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Naive approach

Control $\operatorname{Poly}^{(1)}(\langle A; F\rangle^n)$ by $F^{(\ell)}$ for high $\ell \in \mathbb{N}_+$.

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Putting it together

Lemma

If $\mathbf{A}$ *and* $\mathbf{B}$ *are algebras on* A *and*
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$\implies \left|\pi^{\mathbf{A}}\right| \leq t$.

Connection to cdeg (k)

Suppose $|A| = k$.

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Let $F, G \subseteq \mathrm{O}_A$ be clones, put $\mathbf{A} := \langle A; F \rangle$, $\mathbf{B} := \langle A; G \rangle$.

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- $\implies F^* = G^*$.
- $\implies \mathrm{cdeg}\,(k) \leq 1 + k^t$.

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Corollary

Let $F, G \subseteq \mathrm{O}_A$ be clones, $F^{(1+k^t)} = G^{(1+k^t)} \implies F^* = G^*$.
Hence, $\mathrm{cdeg}(k) \leq 1 + k^t$.

The big surprise for the end. . .

The Burris–Willard conjecture holds. . .

Theorem (С. С. Марченков, 2006/2008)

Every clone on a k-element set that is closed w.r.t. primitive positive definitions is primitive positively generated by its k-ary functions.

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. . . provided one allows **disjunction** of function graphs.

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