On simple Taylor algebras

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2-intersection property

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One of the motivations: CSP

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  - Example: rock-paper-scissors 3-element algebra

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Recall: No algebra in $HS(\mathbf{A})$ is a reduct of affine algebra
$\Leftrightarrow$ $HSP(\mathbf{A})$ omits **1** and **2**
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Theorem (BK, conjectured by Valeriote)

A *has the 2-intersection property* $\Leftrightarrow$ **A** *is SD($\wedge$).*

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Recall: $\mathbf{A}$ is Taylor

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$\Leftrightarrow$ ...

The old and the new (result)

Theorem (The old)

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- $\mathbf{A}$ is SD($\wedge$)

Then $\mathbf{A}$ *has the 2-intersection property*

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- $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are SD($\wedge$)
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If

- $\mathbf{A}_1, \ldots, \mathbf{A}_n$ *are Taylor, simple, non-abelian*
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Then $R \cap S \neq \emptyset$

The real result: a rectangularity theorem

Recall: $B \leq \mathbf{A}$ is absorbing if $\mathbf{A}$ has a term operation t with $t(B, B, \ldots, B, A, B, B, \ldots, B) \subseteq B$

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If

- $\mathbf{A}_1, \ldots, \mathbf{A}_n$ *are Taylor, simple, non-abelian*
- $B_1, \ldots, B_n$ *are minimal absorbing subuniverses of* $\mathbf{A}_1, \ldots, \mathbf{A}_n$
- $R \leq_{sd} \mathbf{A}_1 \times \cdots \times \mathbf{A}_n$ *is irredundant*
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- Similar theorem for conservative algebras $\rightarrow$ Dichotomy for conservative CSPs

A consequence: pointing operation

Definition

A a term operation t of $\mathbf{A}$ points to $b \in A$ if
$\exists(a_1, \ldots, a_n) \in A^n$ such that
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Known: $\mathbf{A}$ is SD($\wedge$) $\Leftrightarrow$ every subalgebra of $\mathbf{A}$ has a pointing term operation.

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- Rectangularity theorem $\Rightarrow R = A^k$.

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- if S_b is linked for some b

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  - **Fact:** $R_{12} = \mathbf{A}_1 \times \mathbf{A}_2$ is absorption-free
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  - $\Rightarrow \mathbf{A}_1$ (and $\mathbf{A}_2$) is abelian.

Thank you!