On centralizers of monounary algebras

Danica Jakubíková-Studenovská,
Miroslava Šuličová

P. J. Šafárik University, Košice

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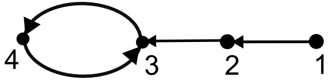
Definition

For a nonempty set A, a mapping $f\colon A \to A$ is called a *unary operation* on A. The pair (A, f) is said to be a *monounary algebra*.

- to each monounary algebra (A, f) there corresponds a directed graph, where the vertex set is A and the edges are ordered pairs $[x, y]$, where $x, y \in A$ and $y = f(x)$



Definition

Let (A, f) be a monounary algebra, $x, y \in A$.
Put $f^0(x) = x$.
If $n \in \mathbb{N}$ and $f^{n-1}(x)$ is defined, then we denote
$f^n(x) = f\left(f^{n-1}(x)\right)$.

For $n \in \mathbb{N}_0$, $f^{-n}(x) = \{z \in A : f^n(z) = x\}$. Denote it by the symbol $\downarrow x$.
Note, $f^{-1}(x)$ is the set of all predecessors of x.

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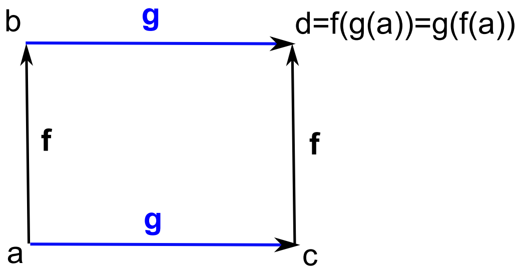
Definition

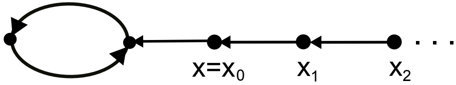
We say that the mappings $f\colon A \to A$, $g\colon A \to A$ *commute* if $f\,(g\,(a)) = g\,(f\,(a))$ for each $a \in A$.



Preliminaries

Definition

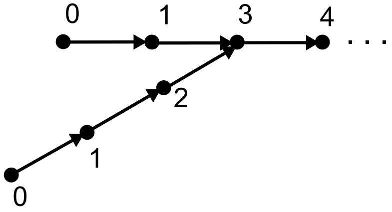
We will denote $s(x) = \infty$ if there exists a sequence $\{x_n\}_{n \in \mathbb{N}_0}$ of elements belonging to A with the property $x_0 = x$ and $f(x_n) = x_{n-1}$ for each $n \in \mathbb{N}$.



Preliminaries

Definition

Next, $s(x) = k$, where $k \in \mathbb{N}_0$, if k is the largest element of $\mathbb{N}_0$ such that $f^{-k}(x) \neq \emptyset$.



Preliminaries

Definition

The *centralizer* of a monounary algebra (A, f) is the set $\mathcal{C}(A, f)$ of those mappings $g \colon A \to A$ which commute with the mapping f.

Definition

The *first centralizer* of (A, f): $\mathcal{C}_1(A, f) = \mathcal{C}(A, f)$.
The *second centralizer* of (A, f) is the set

$$\mathcal{C}_2(A, f) = \bigcap_{g \in \mathcal{C}_1(A,f)} \mathcal{C}_1(A, g).$$

Lemma

Let (A, f) be a monounary algebra. Then $\mathcal{C}_3(A, f) = \mathcal{C}_1(A, f)$.

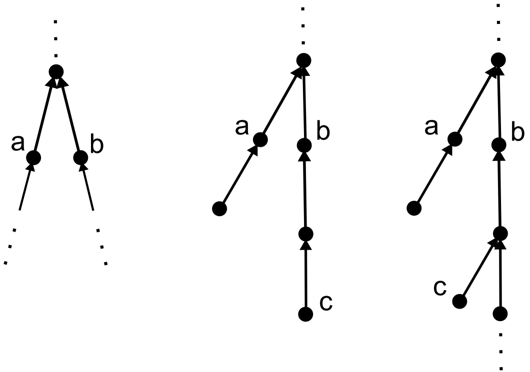
AIM

Describe all *connected* monounary algebras (A, f) with $\mathcal{C}_2(A, f) = \mathcal{C}_1(A, f)$.

Necessary conditions for connected monounary algebras

Lemma

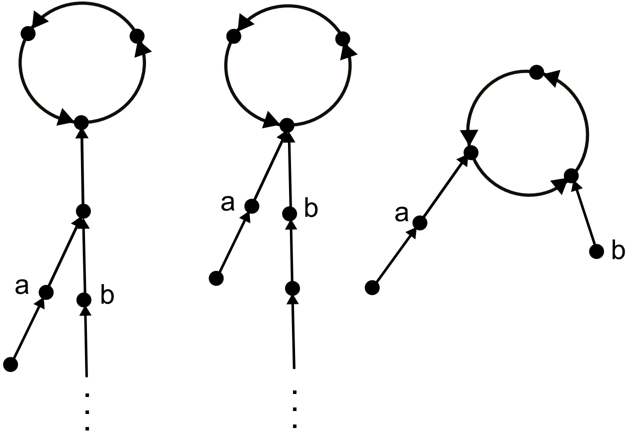
Let $a, b \in A$ be distinct non-cyclic elements such that $f(a) = f(b)$. Suppose that either $s(a) = s(b) = \infty$ or $\infty \neq s(a) \leq s(b) \in \mathbb{N}_0 \cup \{\infty\}$ and there exists $c \in \downarrow b$ such that $s(c) = 0$. Then $\mathcal{C}_2(A, f) \neq \mathcal{C}_1(A, f)$.



Necessary conditions for connected monounary algebras

Lemma

Suppose that (A, f) contains a cycle. Let $a, b \in A$ be distinct non-cyclic elements such that $f(a) = f(b)$ or $f(a)$ and $f(b)$ lie on the cycle. Then $\mathcal{C}_2(A, f) \neq \mathcal{C}_1(A, f)$.



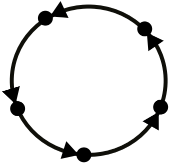
A result for connected monounary algebras

Theorem

Let (A, f) be a connected monounary algebra. Then $\mathcal{C}_2(A, f) = \mathcal{C}_1(A, f)$ if and only if (A, f) is isomorphic to one of the following algebras:

(a) (A, f) is a cycle,

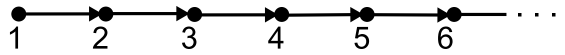
$\mathbb{Z}_n$



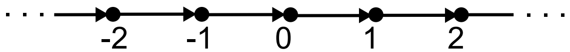
A result for connected monounary algebras

(b) $A = \mathbb{N}$ ($A = \mathbb{Z}$), where $f(n) = n + 1$ for each $n \in \mathbb{N}$ ($n \in \mathbb{Z}$),

$\mathbb{N}$



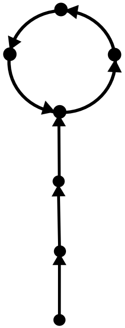
$\mathbb{Z}$



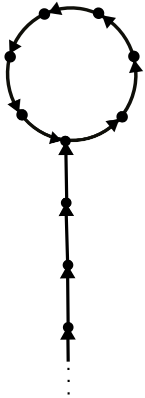
A result for connected monounary algebras

(c) (A, f) contains a cycle C, and there exists the only element $x \in C$ such that $\left|f^{-1}(x)\right| = 2$ and for each $d \in A \setminus \{x\}$ is $\left|f^{-1}(d)\right| \leq 1$,

$L_{n,k}$ (for $k \in \mathbb{N}$)



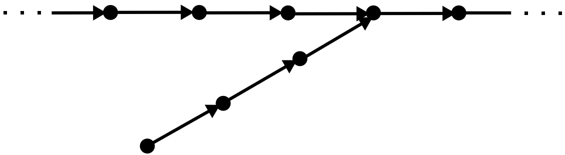
$L_{n,\infty}$



A result for connected monounary algebras

(d) (A, f) does not contain a cycle, there exists the only element $c \in A$ such that $\left|f^{-1}(c)\right| = 2$ and $\left|f^{-1}(d)\right| \leq 1$ for all $d \in A \setminus \{c\}$, and there exists the only subalgebra (B, f) of algebra (A, f) such that $(B, f) \cong (\mathbb{Z}, f)$.

$L_{\infty,k}$



AIM

Describe all *non-connected* monounary algebras (A, f) with $\mathcal{C}_2(A, f) = \mathcal{C}_1(A, f)$.

Excluded cases

Lemma

If B is a connected component of (A, f) with $C_1(A, f) = C_2(A, f)$, then $C_1(B, f) = C_2(B, f)$.

Lemma

A non-connected monounary algebra (A, f) must not contain two one-element cycles.

Lemma

If $\varphi \in C_1(A, f)$ and there exist different components B_1, B_2 of (A, f) such that $\varphi(B_1) \subseteq B_2$, then $C_1(A, f) \neq C_2(A, f)$ or there is a one-element cycle $\{c\}$ in B_2 such that $\varphi(b) = c$ for all $b \in B_1$.

A result for non-connected monounary algebras

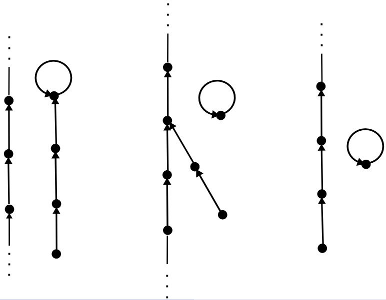
Theorem

Let (A, f) be a non-connected monounary algebra. Then $\mathcal{C}_2(A, f) = \mathcal{C}_1(A, f)$ if and only if there exists a component B of (A, f) such that exactly one of the following conditions is satisfied:

(a) $B \cong \mathbb{Z}$ and $A \setminus B \cong L_{1,k}$ for some $k \in \mathbb{N}_0$,

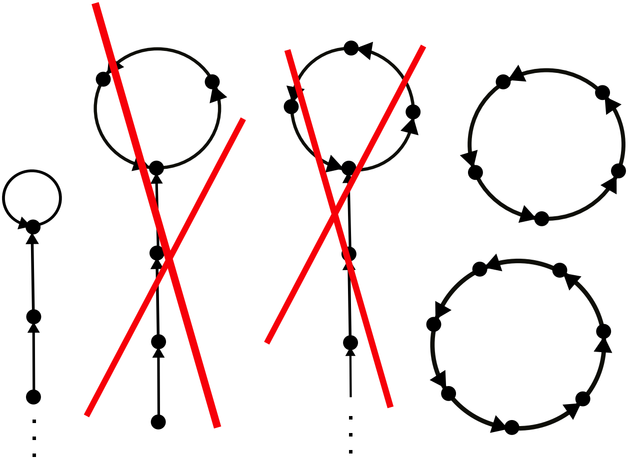
(b) $B \cong L_{\infty,k}$, where $k \in \mathbb{N}$, and $A \setminus B \cong \mathbb{Z}_1$,

(c) $B \cong \mathbb{N}$ and $A \setminus B \cong \mathbb{Z}_1$,

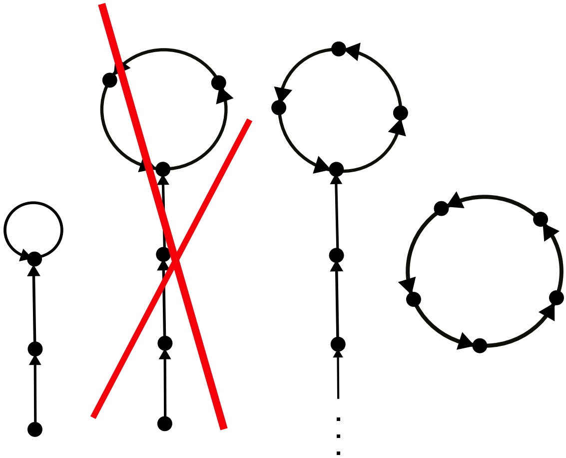


A result for non-connected monounary algebras

(d) $B \cong L_{1,\infty}$ and the system of connected components of $A \setminus B$ is isomorphic to $\{\mathbb{Z}_{n_i}\}_{i \in I}$, where $n_i \nmid n_j$ for $i, j \in I$, $i \neq j$,

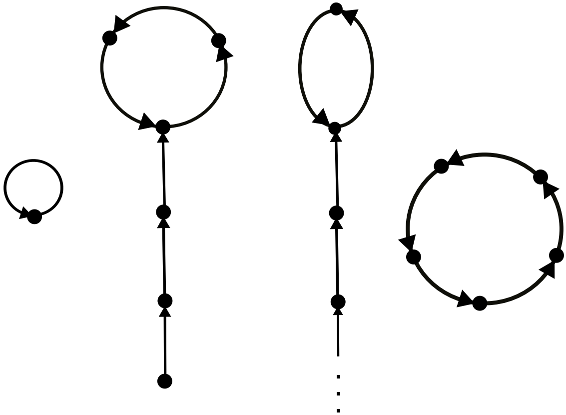


(e) $B \cong L_{1,m}$ for some $m \in \mathbb{N}$ and the system of connected components $A \setminus B$ is isomorphic to $\{L_{n_i,k_i}\}_{i\in I}$, $k_i \in \{0,\infty\}$ for $i \in I$, where $n_i \nmid n_j$ for $i,j \in I$, $i \neq j$,



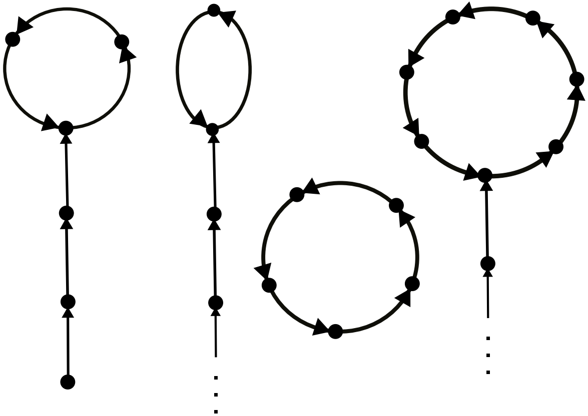
A result for non-connected monounary algebras

(f) $B \cong \mathbb{Z}_1$ and the system of connected components $A \setminus B$ is isomorphic to $\{L_{n_i,k_i}\}_{i \in I}$, $k_i \in \mathbb{N}_0 \cup \{\infty\}$ for $i \in I$, where $n_i \nmid n_j$ for $i, j \in I$, $i \neq j$,



A result for non-connected monounary algebras

(g) the system of connected components of (A, f) is isomorphic to $\{L_{n_i,k_i}\}_{i\in I}$, $k_i \in \mathbb{N}_0 \cup \{\infty\}$ for $i \in I$, where $n_i \nmid n_j$ for $i, j \in I$, $i \neq j$.



Thank you for your attention.