Endomorphism spectrum of finite monounary algebras

D. Jakubíková-Studenovská
P. J. Šafárik University, Košice, Slovakia

coauthor K. Potpinková

Summer School on Algebra and Ordered Sets
Stará Lesná September 6–12, 2014

Introduction

- One of the most important tools in studying universal algebra is the notion of **endomorphism**.
- A rather large series of further algebraic notions is based on endomorphisms.
- The specific sign of an algebraic structure $\mathcal{A}$ we will be interested in, is

  the **endomorphism spectrum** $\operatorname{spec}\mathcal{A}$ of $\mathcal{A}$.

A monounary algebra and the corresponding graph

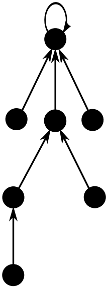
Let (A, f) be a monounary algebra

- to (A, f) there corresponds a **directed graph**

$$G(A, f) = (A, E)$$

  such that $\quad E = \{(a, b) \in A \times A : f(a) = b\}$

- note that $G(A, f)$ may contain loops



A monounary algebra and the corresponding (quasi)ordered set

Let (A, f) be a monounary algebra

- assign to it a binary relation π defined as follows: for $a, b \in A$,

$$a \pi b \Leftrightarrow \exists n \geq 0 : f^n(a) = b$$

- the relation π is a quasiorder, in general.

Fact: This relation is a **partial order** if and only if (A, f) contains no cycle of length at least two.
Such monounary algebras are said to be **acyclic**, and if also connected, then **rooted**.

Basic notion

Definition

Let $\mathcal{A}$ be an algebraic structure. The **endomorphism spectrum** of $\mathcal{A}$ ($\operatorname{spec}\mathcal{A}$, shortly) is the set of all positive integers k such that there exists an endomorphism h of $\mathcal{A}$ having the property that the image $\operatorname{Im} h$ has k elements:

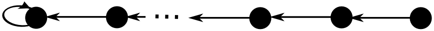
$$\operatorname{spec}\mathcal{A} = \{|\operatorname{Im} h| : h \in \operatorname{End}\mathcal{A}\}$$

The notion of endomorphism spectrum of an algebraic structure was introduced by K. Grant, R. J. Nowakowski and I. Rival (1995) for the case of finite **partially ordered sets**.

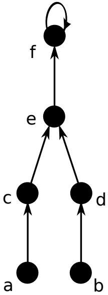
- Choosing from their results:
  for any positive integer k and any "*enough large*" partially ordered set P, $\operatorname{spec} P \supseteq \{1, 2, \ldots, k\}$.
  An analogous assertion for monounary algebras fails to hold.

- It is easy to show that the endomorphism spectrum of the partially ordered set corresponding to an acyclic monounary algebra *includes* the whole spectrum of the monounary algebra: $\operatorname{spec}(A, \leq) \supseteq \operatorname{spec}(A, f)$.
  The converse implication is not valid in general.

Examples



- $\text{spec}(A, f) = \{1, 2, \ldots, |A|\} = \text{spec}(A, \leq)$



- $\text{spec}(A, f) = \{1, 2, 3, 4, 6\} \neq \{1, 2, 3, 4, , 5, 6\} = \text{spec}(A, \leq)$

Endomorphism spectra

Definition

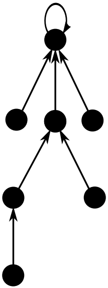
The endomorphism spectrum of a monounary algebra (A, f) is said to be **complete**, if $\mathrm{spec}(A, f) = \{1, 2, \ldots, |A|\}$.

Definition

The term "endomorphism spectrum of algebra (A, f) **skips the set** M", where $M \subseteq \{1, 2, \ldots, |A|\}$ will mean that $\mathrm{spec}(A, f) = \{1, 2, \ldots, |A|\} \setminus M$.
If $M = \{m_1, \ldots, m_k\}$, then we say also that the endomorphism spectrum **skips the numbers** $m_1, \ldots, m_k$.

Definition

A connected partial monounary subalgebra (B, f) of an algebra (A, f) is called a **tail**, if it contains a leaf, $|f^{-1}(a)| = 1$ for each element $a \in B$ which is not a leaf and $|f^{-1}(f(b))| \geq 2$ for the element $b \in B$ of the least height (such b exists and is uniquely determined). Next, $|B|$ is called the length of the tail B.



Some questions

concerning endomorphism spectra, as a contribution to a classification of finite monounary algebras:

- Does for each $S \subset \mathbb{N}$ exist a monounary algebra (A, f) with $\mathrm{spec}(A, f) = S$?
- When does it exist in the case $|S| = 1, 2, \ldots$?
- Does there exist a rooted monounary algebra …?
- Does there exist a *rooted* monounary algebra such that $\mathrm{spec}(A, f)$ skips i consecutive numbers?
- Determine the endomorphism spectrum of some special type of a rooted algebra.

$$\vdots$$

Two-element endomorphism spectrum

Lemma

Let $S = \{1, k\}$, $k \in \mathbb{N}$, $k > 1$. *There exists a monounary algebra* (A, f) *with* $S = \mathrm{spec}(A, f)$.

Lemma

Let $S = \{2, k\}$, $k \in \mathbb{N}$, $k > 2$. *A monounary algebra* (A, f) *with* $S = \mathrm{spec}(A, f)$ *exists if and only if* k *is even.*

Corollary

There exists an infinite (countable) system of finite subsets S *of* $\mathbb{N}$ *such that* S *fails to be an endomorphism spectrum of any monounary algebra* (A, f).

Two-element endomorphism spectrum

Proposition

Let $S = \{m, k\}$*,* $m, k \in \mathbb{N}$*,* $k > m \geq 1$*. A monounary algebra* (A, f) *with* $S = \mathrm{spec}(A, f)$ *exists if and only if either* m *divides* k *or there is a decomposition of* m *into a sum* $m = t_1 + t_2 + \cdots + t_j$ *of positive integers such that*

(i) *if* t_{i_1} *divides* t_{i_2}*,* $1 \leq i_1 \leq i_2 \leq j$*, then* $i_1 = i_2$*,*

(ii) *if* $m \neq k - 1$*, then there is* $1 \leq i_0 \leq j$ *such that* t_{i_0} *divides* $k - m$*.*

Spectrum skipping i consecutive numbers

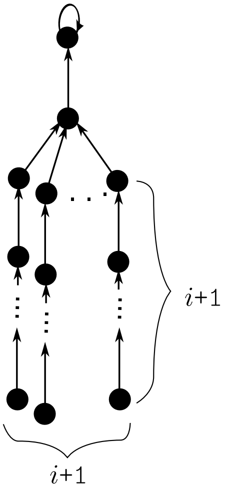
If (A, f) is an algebra of minimal cardinality such that $\text{spec}(A, f)$ skips some i consecutive numbers from the set $\{1, 2, \dots, |A|\}$, $1 \leq i < |A|$, then

- (A, f) contains at least $i + 1$ leaves
- every tail in (A, f) is of length at least $i + 1$

Theorem

A rooted algebra (A, f) with the spectrum skipping some i consecutive numbers from the set $\{1, 2, \dots, |A|\}$, $1 \leq i < |A|$, contains at least $(\mathbf{i} + \mathbf{1})^{\mathbf{2}} + \mathbf{2}$ elements, and this boundary is the **best possible**.

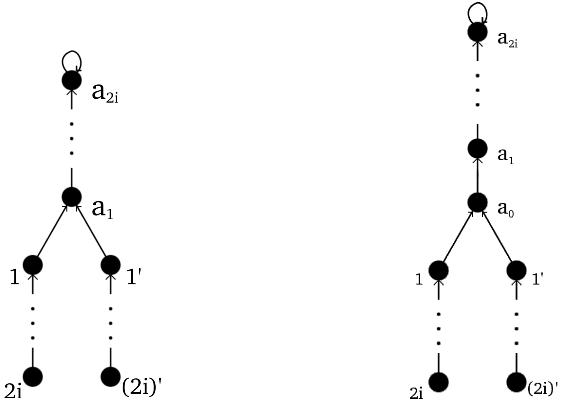
An algebra with spectrum skipping i consecutive numbers



Spectrum skipping *i* consecutive odd (even) numbers

Theorem

For each $i \in \mathbb{N}$ *there exists a rooted algebra* (A, f) *such that* $\mathrm{spec}(A)$ *skips* i *consecutive odd (even, respectively) numbers.*



Also: i consecutive numbers with a given distance k.

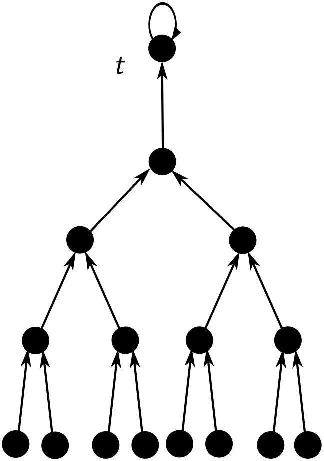
Binary and at least binary trees

Definition

Let (A, f) be a rooted algebra. If $|f^{-1}(a)| \geq 2$ holds for any element which fails to be a leaf, then (A, f) is called **at least binary tree**.

- for a **binary tree**, if a is not a leaf then $|f^{-1}(a)| = 2$.

A binary tree



Binary and at least binary trees

Theorem

The spectrum of any at least binary tree is **complete**.

Thank you for your attention!