Prolific constructions of directed strongly regular graphs with the aid of loops [1]

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Strongly regular graphs

Definition

A simple graph $\Gamma = (V, E)$ is called strongly regular with parameters (n, k, λ, μ), if $|V| = n$ and there exist constants k, λ, μ such that for any $u, v \in V$ the number of uv-walks of length 2 is

1. k, if $u = v$,
2. λ, if $(u, v) \in E$,
3. μ, if $(u, v) \notin E$.

Strongly regular graphs

Let $A = A(\Gamma)$ denote the adjacency matrix of Γ. Then

$$A^2 = k \cdot I + \lambda \cdot A + \mu \cdot (J - I - A),$$

or equivalently,

$$A^2 + (\mu - \lambda) \cdot A - (k - \mu) \cdot I = \mu \cdot J,$$

where I is the identity matrix and J the all-one matrix.

Directed strongly regular graphs

Definition (Duval, 1988)

Let $\Gamma = (V, D)$ be a directed graph, $|V| = n$, in which vertices have constant in- and out-valency k, but now only t edges being undirected ($0 < t < k$). We say that Γ is a directed strongly regular graph with parameters (n, k, t, λ, μ) if there exist constants λ and μ such that the numbers of uw-paths of length 2 are

1. t, if $u = w$;
2. λ, if $(u, w) \in D$;
3. μ, if $(u, w) \notin D$.

Again, equivalently:

$$A^2 = tI + \lambda A + \mu(J - I - A).$$

Directed strongly regular graphs

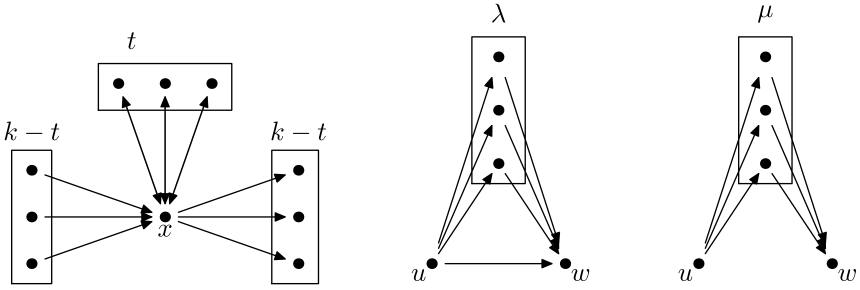

Figure : Locally.

Directed strongly regular graphs

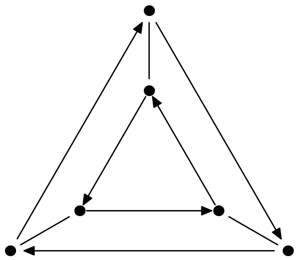

Figure : The smallest DSRG.

The parameter set is $(6, 2, 1, 0, 1)$.

Directed strongly regular graphs

Proposition (Duval, 1988)

If Γ is a DSRG with parameter set (n, k, t, λ, μ) and adjacency matrix A, then the complementary graph $\bar{\Gamma}$ is a DSRG with parameter set $(n, \bar{k}, \bar{t}, \bar{\lambda}, \bar{\mu})$ with adjacency matrix $\bar{A} = J - I - A$, where

$$\begin{aligned} \bar{k} &= n - k + 1 \\ \bar{t} &= n - 2k + t - 1 \\ \bar{\lambda} &= n - 2k + \mu - 2 \\ \bar{\mu} &= n - 2k + \lambda. \end{aligned}$$

Directed strongly regular graphs

Proposition (Ch. Pech, 1997) [Presented in KMMZ]

Let Γ be a DSRG. Then its reverse Γ^T is also a DSRG with the same parameter set.

Definition

We say that two DSRGs Γ_1 and Γ_2 are equivalent, if $\Gamma_1 \cong \Gamma_2$, or $\Gamma_1 \cong \Gamma_2^T$, or $\Gamma_1 \cong \bar{\Gamma}_2$, or $\Gamma_1 \cong \bar{\Gamma}_2^T$; otherwise they are called non-equivalent.

Directed strongly regular graphs

Duval's main theorem

Let Γ be a DSRG with parameters (n, k, t, λ, μ). Then there exists some positive integer d for which the following requirements are satisfied:

$$k(k + (\mu - \lambda)) = t + (n - 1)\mu$$

$$(\mu - \lambda)^2 + 4(t - \mu) = d^2$$

$$d \mid (2k - (\mu - \lambda)(n - 1))$$

$$\frac{2k - (\mu - \lambda)(n - 1)}{d} \equiv n - 1 \pmod 2$$

$$\left| \frac{2k - (\mu - \lambda)(n - 1)}{d} \right| \leq n - 1.$$

Directed strongly regular graphs

Further necessary conditions

$$
\begin{aligned}
0 &\leq \lambda < t &&< k \\
0 &< \mu \leq t &&< k \\
-2(k-t-1) &\leq \mu - \lambda &&\leq 2(k-t).
\end{aligned}
$$

Directed strongly regular graphs

Usually, the main goals concerning DSRG's are:

1. To find a DSRG realizing a new parameter set.
2. To prove a non-existence result.
3. To find an infinite family of DSRG's.

The most important data are collected on the webpage of A. Brouwer and S. Hobart: http://homepages.cwi.nl/~aeb/math/dsrg

Our goals

1. to find DSRGs with new parameter sets
2. to analyze known examples
3. to generalize known examples for infinite series

Sporadic examples

Our methods

1. Cayley digraphs,
2. unions of relations in non-commutative small association schemes.

Using these techniques, we succeeded to construct examples of DSRGs for 28 new parameter sets.

Sporadic examples with new parameter sets

(n, k, t, λ, μ)	(n, k, t, λ, μ)	(n, k, t, λ, μ)
(30,13,11,6,5)	(54,16,12,6,4)	(72,21,15,6,6)
(36,13,7,4,5)	(54,19,9,6,7)	(72,22,9,6,7)
(36,13,11,2,6)	(54,20,16,6,8)	(72,26,10,8,10)
(39,16,12,7,6)	(54,21,17,8,8)	(84,29,19,6,12)
(45,16,8,5,6)	(54,25,14,11,12)	(84,31,17,12,11)
(48,10,6,2,2)	(60,13,5,2,3)	(84,39,27,18,18)
(48,13,7,2,4)	(60,26,20,10,12)	(90,28,16,10,8)
(50,16,10,3,6)	(63,22,10,7,8)	(105,36,16,11,13)
(50,23,13,10,11)	(72,19,11,2,6)	
(54,8,3,2,1)	(72,20,14,4,6)	

Table : We succeeded with 28 parameter sets.

Our special attention has been paid to some graphs on 50 and 72 vertices.

Working with them we conjectured that we have families of DSRGs on $2n^2$ vertices.

Originally these families were approached via so-called block constructions.

A block matrix construction

A few matrices

- P – the $n \times n$ permutation matrix of the cyclic permutation $(1, 2, \ldots, n)$;
- I – the $n \times n$ identity matrix;
- J – the $n \times n$ all-one matrix.
- $$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

A block matrix construction

where

$$A_{11} = A_{22} = \begin{pmatrix} J-I & I & I & \dots & I \\ P & J-I & P & \dots & P \\ P^2 & P^2 & J-I & \dots & P^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P^{n-1} & P^{n-1} & P^{n-1} & \dots & J-I \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} I & P^{n-1} & P^{n-2} & \dots & P \\ I & P^{n-1} & P^{n-2} & \dots & P \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I & P^{n-1} & P^{n-2} & \dots & P \end{pmatrix} \quad A_{21} = \begin{pmatrix} I & P & P^2 & \dots & P^{n-1} \\ I & P & P^2 & \dots & P^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I & P & P^2 & \dots & P^{n-1} \end{pmatrix}.$$

A block matrix construction

Computations show that:

$$A^2 = (2n-1)\cdot I + (n-1)\cdot A + 3\cdot (J - I - A),$$

Theorem A

Matrix A defines a digraph which is a DSRG with parameter set $(2n^2, 3n-2, 2n-1, n-1, 3)$.

A block matrix construction

Let us create matrix A^* by replacing A_{12} with A^*_{12} and A_{21} with A^*_{21}, where

$$A^*_{12} = \begin{pmatrix} I+P & P^{n-1}+I & P^{n-2}+P^{n-1} & \dots & P+P^2 \\ I+P & P^{n-1}+I & P^{n-2}+P^{n-1} & \dots & P+P^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I+P & P^{n-1}+I & P^{n-2}+P^{n-1} & \dots & P+P^2 \end{pmatrix}$$

$$A^*_{21} = \begin{pmatrix} I+P^{n-1} & P+I & P^2+P & \dots & P^{n-1}+P^{n-2} \\ I+P^{n-1} & P+I & P^2+P & \dots & P^{n-1}+P^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I+P^{n-1} & P+I & P^2+P & \dots & P^{n-1}+P^{n-2} \end{pmatrix}.$$

A block matrix construction

Theorem B

The digraph corresponding to the adjacency matrix A^* is a DSRG with parameters $(2n^2, 4n-2, 2n+2, n+2, 6)$.

Due to matrices A_{11} and A_{22} which correspond to lattice square graph, we were aware of the fact, that our DSRGs contain two induced copies of it. On the other hand, lattice square graphs have a nice description in the terms of Latin squares.
Finally, we generalized our constructions for arbitrary Latin squares.

Crucial observations

Observation 1.

Analysis of the groups of automorphisms showed that these graphs can be constructed as Cayley digraphs over the wreath product group $\mathbb{Z}_2 \wr \mathbb{Z}_n$ (of order $2n^2$).

Observation 2.

From the Cayley construction we saw, that replacing $\mathbb{Z}_n$ by any group of order n lead again to a DSRG with the same parameter set. Moreover, for different groups the digraphs were non-isomorphic.

Observation 3.

Explanation of adjacencies in the digraphs using Cayley tables of groups gave us another generalization. Both families of DSRGs could be generalized not only to loops, but even to quasigroups.

Finally, we approached two our general constructions 1 and 3, to be presented on next slides using only language of Latin squares aka quasigroups, respectively, nothing else.

Looking at these construction we realized that similarly we may get DSRGs on $3n^2$ vertices, thus approaching two more constructions labelled further 2 and 4.

Combinatorial structures

Definition

A Latin square of order n is an $n \times n$ array with n different entries, such that each entry occurs exactly once in any row and in any column of the array.

A quasigroup is a set Q with a binary operation "$\cdot$" such that for all $a, b \in Q$ the equations $a \cdot x = b$ and $y \cdot a = b$ have a unique solution in Q.

A loop L is a quasigroup with an identity element $e \in L$ with the property $e \cdot x = x \cdot e = x$ for every $x \in L$.

Construction 1.

Let $(Q, \cdot)$ be an arbitrary quasigroup of order $n \geq 2$.
Define a digraph Γ_1 of order $2n^2$, whose vertex set is
$V(\Gamma_1) = \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \times \mathbb{Z}_2$.
The set $D(\Gamma_1)$ of darts is defined as follows:

- $(x, y, i) \mapsto (z, y, i)$ for all $i \in \mathbb{Z}_2$, $x, y, z \in \{1, 2, \ldots, n\}$, $x \neq z$;
- $(x, y, i) \mapsto (x, z, i)$ for all $i \in \mathbb{Z}_2$, $x, y, z \in \{1, 2, \ldots, n\}$, $y \neq z$;
- $(x, y, 0) \mapsto (xy, z, 1)$ for all $z \in \{1, 2, \ldots, n\}$.
- $(x, y, 1) \mapsto (z, yx, 0)$ for all $z \in \{1, 2, \ldots, n\}$.

Theorem 1.

Γ_1 is a DSRG with parameter set $(2n^2, 3n-2, 2n-1, n-1, 3)$.

Construction 2.

Let $(Q, \cdot)$ be an arbitrary quasigroup of order $n \geq 2$.
Define a digraph Γ_2 of order $3n^2$, whose vertex set is
$V(\Gamma_2) = \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \times \mathbb{Z}_3$.
The set $D(\Gamma_2)$ of darts is defined as follows:

- $(x, y, i) \mapsto (z, y, i)$ for all $i \in \mathbb{Z}_3$, $x, y, z \in \{1, 2, \ldots, n\}$, $x \neq z$;
- $(x, y, i) \mapsto (x, z, i)$ for all $i \in \mathbb{Z}_3$, $x, y, z \in \{1, 2, \ldots, n\}$, $y \neq z$;
- $(x, y, i) \mapsto (xy, z, i+1)$ for all $i \in \mathbb{Z}_3$, and $z \in \{1, 2, \ldots, n\}$.
- $(x, y, i) \mapsto (z, yx, i-1)$ for all $i \in \mathbb{Z}_3$, and $z \in \{1, 2, \ldots, n\}$.

Theorem 2.

Γ_2 is a DSRG with parameter set $(3n^2, 4n-2, 2n, n, 4)$.

Construction 3.

Let $(L, \cdot)$ be an arbitrary loop of order $n \geq 2$, and c any non-identity element of L.
Define a digraph Γ_3 of order $2n^2$, whose vertex set is
$V(\Gamma_3) = \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \times \mathbb{Z}_2$.
The set $D(\Gamma_3)$ of darts is defined as follows:

- $(x, y, i) \mapsto (z, y, i)$ for all $i \in \mathbb{Z}_2$, $x, y, z \in \{1, 2, \ldots, n\}$, $x \neq z$;
- $(x, y, i) \mapsto (x, z, i)$ for all $i \in \mathbb{Z}_2$, $x, y, z \in \{1, 2, \ldots, n\}$, $y \neq z$;
- $(x, y, 0) \mapsto (xy, z, 1)$ for all $z \in \{1, 2, \ldots, n\}$.
- $(x, y, 1) \mapsto (z, yx, 0)$ for all $z \in \{1, 2, \ldots, n\}$.
- $(x, y, 0) \mapsto (c(xy), z, 1)$ for all $z \in \{1, 2, \ldots, n\}$.
- $(x, y, 1) \mapsto (z, (yx)c, 0)$ for all $z \in \{1, 2, \ldots, n\}$.

Theorem 3.

Γ_3 is a DSRG with parameter set $(2n^2, 4n - 2, 2n + 2, n + 2, 6)$.

Construction 4.

Let $(L, \cdot)$ be an arbitrary loop of order $n \geq 2$, and c any non-identity element of L.
Define a digraph Γ_4 of order $3n^2$, whose vertex set is
$V(\Gamma_4) = \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \times \mathbb{Z}_3$.
The set $D(\Gamma_4)$ of darts is defined as follows:

- $(x, y, i) \mapsto (z, y, i)$ for all $i \in \mathbb{Z}_3$, $x, y, z \in \{1, 2, \ldots, n\}$, $x \neq z$;
- $(x, y, i) \mapsto (x, z, i)$ for all $i \in \mathbb{Z}_3$, $x, y, z \in \{1, 2, \ldots, n\}$, $y \neq z$;
- $(x, y, i) \mapsto (xy, z, i + 1)$ for all $z \in \{1, 2, \ldots, n\}$.
- $(x, y, i) \mapsto (z, yx, i - 1)$ for all $z \in \{1, 2, \ldots, n\}$.
- $(x, y, i) \mapsto (c(xy), z, i + 1)$ for all $z \in \{1, 2, \ldots, n\}$.
- $(x, y, i) \mapsto (z, (yx)c, i - 1)$ for all $z \in \{1, 2, \ldots, n\}$.

Theorem 4.

Γ_4 is a DSRG with parameter set $(3n^2, 6n - 2, 2n + 6, n + 6, 10)$.

Proof of Theorems 1-4. (outline)

- existence of k and t;
- existence of λ and μ:
  - counting over darts and non-darts;
  - various types of directed paths of length 2;
  - uniqueness of solutions of equations $x \cdot a = b$ and $a \cdot y = b$.

Automorphism group of Γ_1 for the group case

Theorem 5.

If Γ_1 from Construction 1 is arising from a group K, then for its full group G of automorphisms holds:

$$G \cong (K^2 \rtimes \mathrm{Aut}(K)) \rtimes S_2.$$

Remark

The proof follows from the classical results about the automorphism groups of 3-nets, corresponding to group Latin squares (see e.g. survey of Heinze and Klin).

Isotopism and isomorphism of quasigroups

Two quasigroups $(Q_1, \cdot)$ and $(Q_2, \circ)$ are isomorphic,
if there exists a bijection f from Q_1 to Q_2 such that
for all $a, b, c \in Q_1$: $(a \cdot b = c) \iff (a^f \circ b^f = c^f)$.

Two Latin squares L_1, L_2 represented as $n \times n$-arrays are isotopic,
if there exist three permutations $h_1, h_2, h_3 \in S_n$ such that the action
h_1 on rows, h_2 on columns, h_3 on symbols of L_1 brings L_1 to L_2.

Numbers of DSRGs

Considering loops and quasigroups of order n, our four constructions give the following number of non-isomorphic DSRGs:

n	ISOTC	ISOMC	Constr.1.	Constr.2.	Constr.3	Constr.4
3	1	1	1	1	1	1
4	2	2	2	2	2	2
5	2	6	3	6	9	10
6	22	109	38	109	341	365

Table : Numbers of different combinatorial objects

ISOTC = nr. of isotopy classes of loops

ISOMC = nr. of isomorphism classes of loops

Conjectures

Conjecture 1.

The number of non-isomorphic DSRG's from our constructions grows exponentially over n.

Conjecture 2.

The number of non-isomorphic DSRG's from our Construction 2 is equal to the number of isomorphism classes of loops.

Computer tools

- GAP (Group Algorithm and Programming)
- COCO II (unpublished version by S. Reichard)
- GRAPE (L.H. Soicher)
- nauty (B.D. McKay)
- LOOPS (G.P. Nagy, P. Vojtěchovský)

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Thank you

Thank you for your attention.