On monounary algebras with easy direct limits

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Outline

1. Introduction
2. Examples
3. On upward directed poset
4. On linearly ordered set

an algebra with easy direct limits	=	an algebra from which we can obtain by a direct limit construction a retract of itself only

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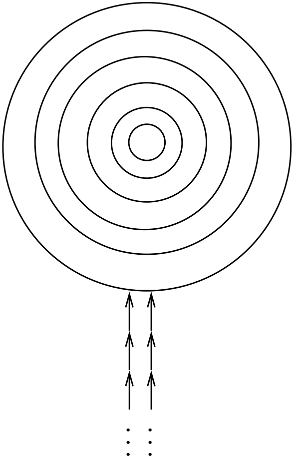
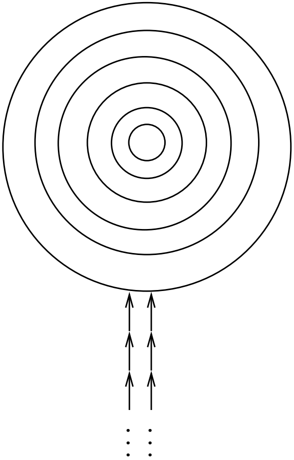
Theorem (1999 H., Ploščica)

Every finite algebra is an algebra with easy direct limits.



Theorem (SSAOS 2013)

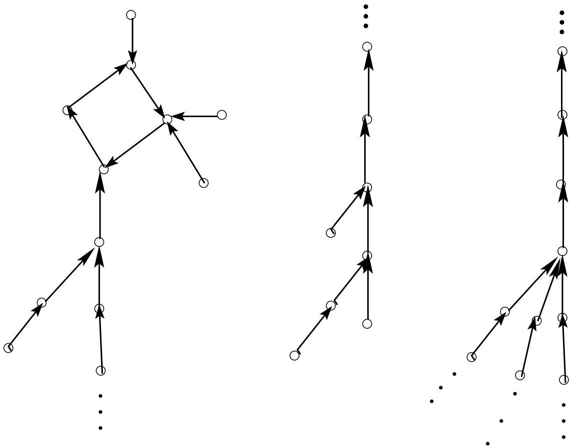
There exists a set of nonisomorphic monounary algebras with easy direct limits which has the cardinality of continuum.



Theorem (SSAOS 2014)

The class of all nonisomorphic monounary algebras with easy direct limits is the set of the cardinality of continuum.

Example 1



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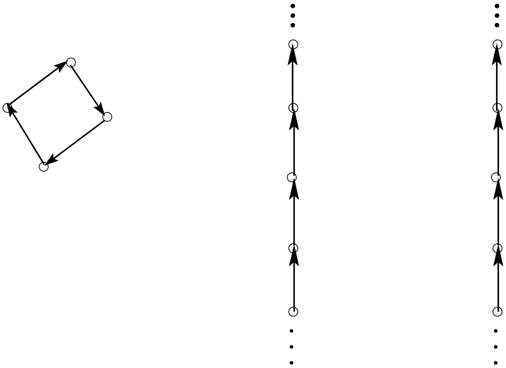
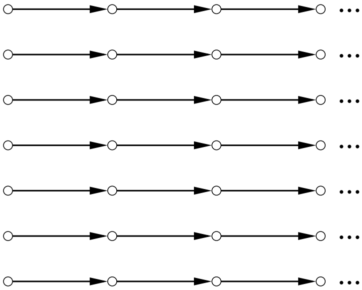


Example 2



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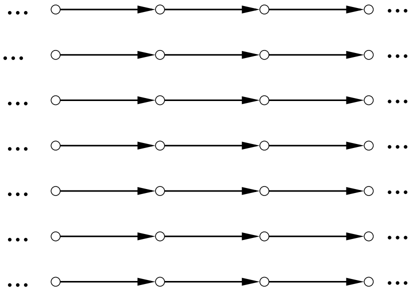

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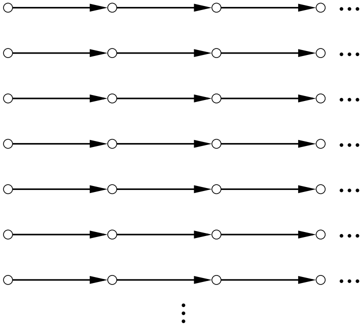

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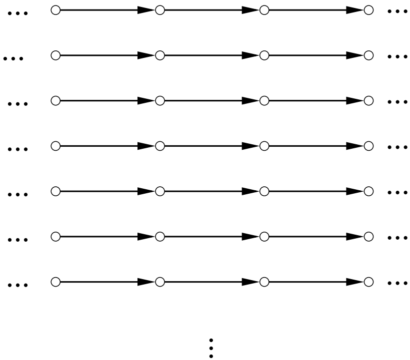

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f .... a mapping from A into A

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Proposition

Let (A, f) be a monounary algebra with easy direct limits. Then

1. $f^{-1}(a)$ *is finite for each* $a \in A$*,*
2. *every component of* (A, f) *without a cycle contains a chain,*
3. $m, k_i \in \mathbb{N}$ *for every* $i \in \mathbb{N}$*.*

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Corollary

Every monounary algebra with easy direct limits is countable.

Theorem

Let f be a bijection. Then TFAE:

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Lemma

There exists a monounary algebra with easy direct limits such that this algebra has uncountable many nonisomorphic retracts.

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m_2 .... the number of components of (A, f) without a cycle which contain a chain

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Proposition

Let (A, f) be a monounary algebra with easy linearly direct limits. Then

1. *there does not exist $a \in A$ and $\{d_k, k \in \mathbb{N}\} \subseteq f^{-1}(a)$ such that $degree(d_k) \in \mathbb{N}$ and $degree(d_k) <\ degree(d_{k+1})$ for every $k \in \mathbb{N}$,*
2. *there exists a retract (B, g) of (A, f) such that g is surjective,*
3. *if $m_2 \in \mathbb{N}$, then $m_1 = 0$, else $m_1 \leq m_2$.*

Lemma

There exists a set of nonisomorphic monounary algebras with easy linearly direct limits which has the cardinality of continuum.

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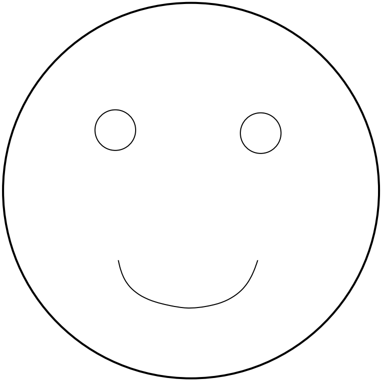
Theorem (??? SSAOS 2015)

??? The class of all nonisomorphic monounary algebras with easy linearly direct limits is... ???



Definition

A direct system of algebras $\{I, A_i, \varphi_{ij}\}$ contains

1. upward directed poset $\langle I, \leq \rangle$, $I \neq \emptyset$;
2. algebra (A_i, F) for each $i \in I$;
3. homomorphism φ_{ij} of A_i into A_j $(i < j)$;
   φ_{ii} the identity on A_i;
   $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$ $(i < j < k)$.

$x \equiv y$ if $\varphi_{ik}(x) = \varphi_{jk}(y)$

The direct limit of $\{I, A_i, \varphi_{ij}\}$ is $(\overline{A}, F)$, where

$\overline{A} = \dot{\bigcup}_{i \in I} A_i / \equiv$

$f(\overline{x}_1, \overline{x}_2, ..., \overline{x}_n) = \overline{f(\varphi_{i_1 k}(x_1), ..., \varphi_{i_n k}(x_k))}$