Direct product of l–algebras and Unification. Applications to Residuated Lattices.

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Summer School on General Algebra and Ordered Sets, SSAOS
Stara Lesna (High Tatras), 6– 12. 09.2014

Summary:

we give sufficient algebraic conditions for a variety $\mathcal{V}$ of l-algebras (which are based on lattices) of the kind $\mathcal{A} = (L, \wedge, \vee, 0, 1, F)$ such that:

finitely presented and projective algebras from the variety $\mathcal{V}$ are closed under finite direct product.

It follows from this condition that unification in $\mathcal{V}$ is filtering (i.e. for every two unifiers there exists a unifier that is more general then both of them), and hence unification in $\mathcal{V}$ is either unitary or nullary.

UNIFICATION of terms (polynomials) in an equational theory E.

Two terms $t_1(x_1, \ldots, x_n), t_2(x_1, \ldots, x_n)$ in variables $x_1, \ldots, x_n = \underline{x}$.
$T(\underline{x})$ - all terms in variables $x_1, \ldots, x_n$,

A substitution $\sigma : \{x_1, \ldots, x_n\} \to T(\underline{y})$ is an $E-$ *unifier* for t_1, t_2 if

$\vdash_E \sigma t_1 = \sigma t_2$.

In this case t_1, t_2 - *unifiable* in E. Solving equations: $t_1 = t_2$ in E.

Given two E-unifiers σ, $\tau : \underline{x} \to T(\underline{y})$ for t_1, t_2,
σ is *more general than* τ, $\tau \preceq \sigma$,
if there is a substitution θ such that, for $x \in \underline{x}$,

$$\vdash_E \theta(\sigma(x)) = \tau(x).$$

$\preceq$ is a preorder (reflexive, transitive).

A *mgu, most general unifier* for t_1, t_2 in E is a $E-$unifier σ for t_1, t_2 such that σ is more general then any E-unifier for t_1, t_2.

Unification Types of equational theories (varieties):

Each equational theory (variety) may have one of the four types:

unitary (or **1**), each unifiable t_1, t_2 has a mgu (best),
finitary (or ω), each unifiable t_1, t_2 - finitely m. max (w.r.t. $\preceq$) unifiers, and not unitary,
infinitary (or ∞), each unifiable t_1, t_2 - infin. m. max unifiers, and neither unitary nor finitary (roughly),
nullary, (or **0**), for some unifiable t_1, t_2 max unifiers do not exist (worst)

If unification in E is unitary (or finitary) and decidable, then there are applications of some deduction technique to Automated Theorem Provers, industrial databases, Description Logic etc.
Not applicable if unification in E is infinitary or nullary.

Applications in logic: admissibility of inference rules.

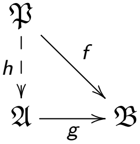
semigroups	infinitary	Plotkin 1972
commutat. semigroups	finitary	Livesey, Siekmann 19
groups	infinitary	Lawrence 1989
Abelian groups	finitary	Lankford 1979
rings	infinitary	Lawrence 1987
commutat. rings	nullary or infinitary	Burris, Lawrence 198
lattices (distr.latt.)	nullary	Willard 1991
semilattices	finitary	Livesey, Siekmann 19
Boolean algebras	unitary	Biitner, Simonis 1987
discriminator algebras	unitary	Burris 1987
Heyting algebras	finitary	Ghilardi 1999
closure (interior) alg.	finitary	Ghilardi 2000
equivalential algebras	unitary	Wroński 2005
q-linear closure (int.) a.	unitary	Dzik, Wojtylak 2011
Fregean varieties	unitary	Slomczynska 2011
MV- algebras (Łukas.)	nullary	Marra, Spada 2011
var. distr. pseudocpl. latt.	nullary	Cabrer 2013

See Stan Burris http://www.math.uwaterloo.ca/ $\sim$ snburris/

Algebraic approach by Ghilardi (1999)

An algebra $\mathfrak{B} \in \mathcal{V}_E$ is *finitely presented*, if there is a finite set of variables, $x_1, \dots, x_k = \underline{x}$ and a finite set S of equations of terms with variables in $\underline{x}$ such that $\mathfrak{B}$ is isomorphic to a quotient algebra $\mathcal{F}_E(\underline{x})/_{\sim}$, where $\sim$ is a congruence defined as follows :
$t_1 \sim t_2$ iff $S \vdash_E t_1 = t_2$

An algebra $\mathfrak{P}$ in a variety $\mathcal{V}$ is *projective* in $\mathcal{V}$ if for every $\mathfrak{A}$, $\mathfrak{B}$ of $\mathcal{V}$ and homomorphisms $f : \mathfrak{P} \to \mathfrak{B}$, $g : \mathfrak{A} \to \mathfrak{B}$ (where g is epi) there is a $h : \mathfrak{P} \to \mathfrak{A}$ such that the following diagram commutes



$\mathfrak{P}$ is projective in $\mathcal{V}$ iff $\mathfrak{P}$ is a *retract of a free algebra* in $\mathcal{V}$, i.e. there are q and m such that

$$\mathfrak{P} \xrightarrow[m]{} \mathcal{F}_V(\underline{x}) \xrightarrow[q]{} \mathfrak{P}$$

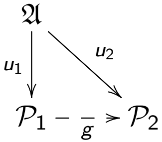
and $q \circ m = 1$ (q is onto, m is 1-1).

Ghilardi: E-unification problem corresponds to a finitely presented algebra $\mathfrak{A} \in V_E$.

A *unifier (a solution)* for $\mathfrak{A}$ is a pair given by: a projective algebra $\mathcal{P}$ and a homomorphism $u : \mathfrak{A} \to \mathcal{P}$.

$\mathfrak{A}$ is *unifiable* if there is a unifier for it.

Given two unifiers u_1 and u_2 for $\mathfrak{A}$, $u_1 : \mathfrak{A} \to \mathcal{P}_1$ is *more general* then $u_2 : \mathfrak{A} \to \mathcal{P}_2$, $u_2 \preceq u_1$, if there is a homomorphism g such that the following diagram commutes:



Unification types in symbolic and algebraic approach coincide.

Lack of characterizations of unification types.

(S.Burris) Discriminator varieties have unitary unification.

Unification in L is *filtering* if, for every two unifiers there exists a unifier that is more general then both of them (then type 1 or 0).

Ghilardi (2003): If a modal logic L contains $K4$, then unification in L is filtering $\iff \Diamond^+\Box^+x \to \Box^+\Diamond^+x \in L$

WD(2006): If a logic L extends intuitionistic logic, then unification in L is filtering $\iff \neg x \vee \neg\neg x \in L$.

Ghilardi (2003) Unification in L is filtering iff the *direct product of two finitely presented and projective algebras from $\mathcal{V}_L$ is finitely presented and projective*.

Algebraic preliminaries:

Let L be a bounded lattice with 0 and 1.
$a \in L$ is called a *central element* of L if a is complemented and for all $x, y \in L$ the sublattice generated by $\{a, x, y\}$ is distributive.

Cen(L) - all central elements, a Boolean sublattice of the lattice L,
$c \in$ Cen(L) has a single complement $\overline{c} \in$ Cen(L);
the pair $\{c,\overline{c}\}$ is called a *central pair* of L

$c \in$ Cen(L), induces a congruence

$$\theta_c = \{(x, y) \mid x \wedge c = y \wedge c\}.$$

for any $c \in$ Cen(L), θ_c and $\theta_{\overline{c}}$ form a factor congruence pair of L;
conversely, if θ_1 and θ_2 are factor congruences of a bounded lattice L: $L \cong L/\theta_1 \times L/\theta_2$, then there exists a $c \in$ Cen(L) such that $\theta_1 = \theta_c$, $\theta_2 = \theta_{\overline{c}}$.

An algebra $\mathcal{A} = (L, \wedge, \vee, 0, 1, F)$ is called an *l-algebra* if $(L, \wedge, \vee, 0, 1)$ is a bounded lattice, and any n-ary term $f : L^n \to L$ of it is *centre-preserving,* that is, for every $c \in \mathrm{Cen}(L)$,

$$(x_i, y_i) \in \theta_c,\ i = 1, ..., n \text{ implies } (f(x_1, ..., x_n), f(y_1, ..., y_n)) \in \theta_c.$$

Examples of l-algebras: bounded lattices, p-algebras, ortholattices, Heyting algebras etc., also residuated lattices (we will show it)

- any l-algebra is congruence distributive,
- the factor congruences of an l-algebra and of its underlying lattice L coincide.

A congruence θ of Con$\mathcal{A}$ is said to be *compact* if for every (nonempty) $\Phi \subseteq$ Con$\mathcal{A}$, $\theta \leq \bigvee \Phi$ implies $\theta \leq \bigvee F$, for some finite nonempty $F \subseteq \Phi$.
An algebra $\mathcal{A} = (A, F)$ with a constant 1 is called 1*-regular (or weakly regular)*, if for each $\varphi, \theta \in$ Con$\mathcal{A}$,
$[1]_\varphi = [1]_\theta$ implies $\varphi = \theta$.

A variety $\mathcal{V}$ with a constant 1 is 1-regular if each $\mathcal{A} \in \mathcal{V}$ is 1-regular. Any congruence φ of a 1-regular algebra $\mathcal{A}$ is generated by its congruence class $[1]_\varphi$, i.e. $\varphi = \theta([1]_\varphi)$.

Theorem.(B. Csákány) *A variety $\mathcal{V}$ with a constant 1 is 1-regular if and only if there exist binary terms $d_1(x, y), \dots, d_n(x, y)$ such that*
$d_i(x, x) = 1$, for $i =, \dots, n$ and
$d_1(x, y) = 1, \dots, d_n(x, y) = 1$ implies $x = y$.

Any compact congruence of $\mathcal{A}$ is a principal congruence and it is generated by a principal filter of L.
$\mathcal{F}_V(\underline{x})$ denoted the free algebra generated by $\underline{x}$

An algebra is finitely presented if it is isomorphic to a finitely generated free algebra divided by a compact congruence.

Corollary. *Let $\mathcal{V}$ be a variety of algebras of the form $\mathcal{A} = (L, \wedge, \vee, 0, 1, F)$ which is 1-regular. Then a finitely presented algebra of $\mathcal{V}$ can be represented (up to isomorphism) by a quotient $\mathcal{F}_V(\underline{x})/\theta[t)$, where $t = 1$ corresponds to equations in the presentation S that define the congruence $\sim$ in $\mathcal{F}_V(\underline{x})/\sim$, in other words: $\theta[t) = \theta(t, 1) = \sim$ in $\mathrm{Con}\mathcal{F}_V(\underline{x})$.*

I. Finitely presented algebras closed under the fin. direct product.

Assume: $\mathcal{V}$ - a variety of (lattice based) l-algebras of the form $\mathcal{A} = (L, \wedge, \vee, 0, 1, F)$ and the following condition holds in $\mathcal{V}$:

(A) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice and any $u, v \in L$ which are complements of each other in L are central elements in L,

Lemma 5. *If $\mathcal{V}$ satisfy the above, then for any $a, b \in L$ with $a \wedge b = 0$ we have*

$$\theta(a, 1) \wedge \theta(b, 1) = \theta(a \vee b, 1)$$

and the congruences $\theta(a, 1)/\theta(a \vee b, 1)$ and $\theta(b, 1)/\theta(a \vee b, 1)$ form a factor congruence pair of the factor algebra $\mathcal{A}/\theta(a \vee b, 1)$.

In addition to (A) we have
(B) Each algebra $\mathcal{A} \in \mathcal{V}$ has a unary term g such that for $v \in L$, $v \wedge g(v) = 0$ and $g(0) = 1$,

Theorem 6. *Let $\mathcal{V}$ be a 1-regular variety of l-algebras satisfying the conditions (A) and (B). If $\mathcal{A}, \mathcal{B} \in \mathcal{V}$ are finitely presented algebras, then their direct product $\mathcal{A} \times \mathcal{B}$ is also finitely presented.*

II. Finitely presented projective algebras closed under finite direct product

(C) Each algebra $\mathcal{A} = (L, \wedge, \vee, 0, 1, F) \in \mathcal{V}$ has two unary terms h and $\urcorner h$, such that for every $v \in L$, $h(v) \wedge \urcorner h(v) = 0$, $h(v) \vee \urcorner h(v) = 1$ and $h(0) = \urcorner h(1) = 1$.

Theorem 8. *Let $\mathcal{V}$ be a 1-regular variety of l-algebras satisfying the conditions (A), (B) and (C). If $\mathcal{A}, \mathcal{B} \in \mathcal{V}$ are finitely presented projective algebras, then $\mathcal{A} \times \mathcal{B}$ is also a finitely presented projective algebra of $\mathcal{V}$.*

Corollary: *If the conditions (A), (B) and (C) hold in a variety of l-algebras $\mathcal{V}$ then unification in $\mathcal{V}$ is either unitary or nullary.*

Application: filtering unification in varieties of residuated lattices.

An algebra $\mathcal{L} = (L, \wedge, \vee, \odot, \to, 0, 1)$ is called an *integral bounded commutative residuated lattice*, IBCRL, or simply *bounded residuated* lattice, if
(1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;
(2) $(L, \odot)$ is a commutative monoid with unit element 1;
(3) $x \odot y \leq z \Leftrightarrow x \leq y \to z$, for all $x, y, z \in L$.

Theorem 14. *Any bounded residuated lattice is an l-algebra satisfying condition (A).*

Now consider $\mathcal{L} = (L, \wedge, \vee, \odot, \to, 0, 1)$ - a bounded residuated lattice satisfying:
$\neg x \wedge x = 0$, for all $x \in L$, where $\neg x := x \to 0$,

it is always pseudocomplemented: $y \wedge x = 0 \Leftrightarrow y \leq \neg x$

Theorem 19. *Let $\mathcal{V}$ be a variety of integral commutative residuated lattices and assume that the Stone identity*

$$\neg x \vee \neg\neg x = 1$$

holds in $\mathcal{V}$.
Then unification in $\mathcal{V}$ is filtering.

Corollary: *If each of residuated lettices generating a variety $\mathcal{V}$ has no zero divisors, then unification in $\mathcal{V}$ is either unitary or nullary.*
In particular, unification in strict fuzzy logics is either unitary or nullary.

Till now: EDPC, deduction theorem etc. needed,
here: no need of EDPC, non-distributive residuated lattices included

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