On a product of n-ary hyperalgebras

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SSAOS 2014, Stará Lesná, September 6–12, 2014

Let n be a positive integer. By an n-ary *hyperalgebra* we understand a pair $\mathbf{G} = (G, p)$ where $G \neq \emptyset$ is a set, the *carrier* of $\mathbf{G}$, and $p : G^n \to \exp G \setminus \{\emptyset\}$ is a map, the *n-ary hyperoperation* of $\mathbf{G}$.

Let $\mathbf{H} = (H, q), \mathbf{G} = (G, p)$ be a pair of n-ary hyperalgebras. A map $f : H \to G$ is a *homomorphism* of $\mathbf{H}$ into $\mathbf{G}$ if $f(q(x_1, \ldots, x_n)) \subseteq p(f(x_1), \ldots, f(x_n))$. We denote by HYP_n the construct of n-ary hyperalgebras with homomorphisms as morphisms.

The usual symbol $\cong$ is used to denote (the relation of) an isomorphism between n-ary hyperalgebras and, if an n-ary partial algebra $\mathbf{H}$ may be embedded into another one $\mathbf{G}$, we write $\mathbf{H} \preceq \mathbf{G}$.
Given n-ary hyperalgebras $\mathbf{H} = (H, q)$ and $\mathbf{G} = (H, p)$, we put $\mathbf{G} \leq \mathbf{H}$ if $p(x_1, \ldots, x_n) \subseteq q(x_i, \ldots, x_n)$ for every $x_1, \ldots, x_n \in G$.
If p is an n-ary hyperoperation on a set G and $X_1, \ldots, X_n$ are nonempty subsets of G, we put
$p(X_1, \ldots, X_n) = \bigcup\{p(x_1, \ldots, x_n); x_1 \in X_1, \ldots, x_n \in X_n\}$.

Let $\mathbf{G}_i = (G, p_i), i \in I$, be a non-empty family of n-ary hyperalgebras. We define the *direct sum* of the family to be the n-ary hyperalgebra $\sum_{i\in I} \mathbf{G}_i = (G, p)$ where, for every $x_1, \dots, x_n \in G, p(x_1, \dots, x_n) = \bigcup_{i\in I} p_i(x_1, \dots, x_n)$. If the set I is finite, say $I = \{1, \dots, m\}$, then we write $\mathbf{G}_1 \uplus \dots \uplus \mathbf{G}_m$ instead of $\sum_{i\in I} \mathbf{G}_i$.

An n-ary hyperalgebra $\mathbf{G} = (G, p)$ is said to be *idempotent* if, for every $x \in G$, $x \in p(x_1, \dots, x_n)$ whenever $x = x_1 = \dots = x_n$. Let $\mathbf{G} = (G, p)$ be an n-ary hyperalgebra. For every $x_1, \dots, x_n \in G$, we put

$$\bar{p}(x_1, \dots, x_n) = \begin{cases} p(x_1, \dots, x_n) \cup \{x\} & \text{if there is } x \in G \text{ with } x = x_1 = \dots = x_n, \\ p(x_1, \dots, x_n) & \text{if there are } i, j \in \{1, \dots, n\} \text{ with } x_i \neq x_j. \end{cases}$$

The n-ary hyperalgebra $(G, \bar{p})$ is called the *idempotent hull* of $\mathbf{G}$ and is denoted by $\bar{\mathbf{G}}$.

Definition

Let $\mathbf{G}_i = (G_i, p_i), i \in I$, be a non-empty family of n-ary hyperalgebras. The *combined product* of the family is the n-ary hyperalgebra $\bigotimes_{i\in I} \mathbf{G}_i = \sum_{i\in I} \prod_{j\in I} \mathbf{G}_{ij}$ where

$$\mathbf{G}_{ij} = \begin{cases} \bar{\mathbf{G}}_j & \text{if } i = j, \\ \mathbf{G}_j & \text{if } i \neq j. \end{cases}$$

If the set I is finite, say $I = \{1, \dots, m\}$, we write $\mathbf{G}_1 \otimes \ldots \otimes \mathbf{G}_m$ instead of $\bigotimes_{i\in I} G_i$.
For every n-ary hyperalgebra $\mathbf{H} = (H, q)$, the operation of combined product defines an endofunctor $\mathbf{H} \otimes - : HYP_n \to HYP_n$ which assigns to every morphism $f : \mathbf{G}_1 \to \mathbf{G}_2$ in HYP_n the morphism $id_H \times f : \mathbf{H} \otimes \mathbf{G}_1 \to \mathbf{H} \otimes \mathbf{G}_1$.

Remark

Let $\mathbf{G}_i, i \in I$, be a non-empty family of n-ary hyperalgebras. We clearly have $\prod_{i\in I} \mathbf{G}_i \le \bigotimes_{i\in I} \mathbf{G}_i \le \prod_{i\in I} \bar{\mathbf{G}}_i = \bigotimes_{i\in I} \bar{\mathbf{G}}_i$ and $\overline{\bigotimes_{i\in I} \mathbf{G}_i} \le \bigotimes_{i\in I} \bar{\mathbf{G}}_i$. If $\mathbf{G}_i$ is idempotent for every $i \in I$, then all the previous inequalities become equalities. If at most one of the n-ary hyperalgebras $\mathbf{G}_i, i \in I$, is not idempotent, then $\bigotimes_{i\in I} \mathbf{G}_i$ is idempotent.

Theorem

Let $\mathbf{G}_i = (G_i, p_i), i \in I \neq \emptyset$, and $\mathbf{H} = (H, q)$ be n-ary hyperalgebras. Then $\sum_{i\in I} \mathbf{H} \otimes \mathbf{G}_i = \mathbf{H} \otimes \sum_{i\in I} \mathbf{G}_i$.

Definition

An n-ary hyperalgebra (G, p) is called
(a) *medial* if, for every $n \times n$-matrix (a_{ij}) over G, from $x_i \in p(a_{i1}, \ldots, a_{in})$ for each $i = 1, \ldots, n$ and $y_j \in p(a_{1j}, \ldots, a_{nj})$ for each $j = 1, \ldots, n$ it follows that $p(x_1, \ldots, x_n) = p(y_1, \ldots, y_n)$,
(b) *diagonal* if, for every $n \times n$-matrix (a_{ij}) over G, we have
$p(p(a_{11}, \ldots, a_{1n}), \ldots, p(a_{n1}, \ldots, a_{nn})) \cap$
$p(p(a_{11}, \ldots, a_{n1}), \ldots, p(a_{1n}, \ldots, a_{nn})) \subseteq p(a_{11}, \ldots, a_{nn})$.

Medial groupoids were studied by T. Kepka and J. Ježek, medial universal algebras are often called commutative and were investigated, e.g., by L. Klukovits. The idempotent, medial and diagonal n-ary algebras were studied by J. Plonka who proved that they are, up to isomorphisms, the algebras $(X_1 \times ... \times X_n, p)$ where $X_1, ..., X_n$ are sets and the operation p is defined by $p((x_1^1, ..., x_n^1), ..., (x_1^n, ..., x_n^n)) = (x_1^1, x_2^2, ..., x_n^n)$. Idempotent, diagonal and medial groupoids are known as *rectangular bands*.

Example

1. Let $(X, \leq)$ be a partially ordered set with a least element 0 and let A be the set of all atoms of $(X, \leq)$. For every pair of elements $x, y \in X$, put $x * y = \{0\}$ if $x = 0$ or $y = 0$ and $x * y = \{z \in X; z < x,\ z < y$ and $z \in A \cup \{0\}\}$ otherwise. Then $(X, *)$ is a hypergroupoid which is both medial and diagonal.
2. Every unary algebra $\mathbf{G}$ consisting of two-element cycles is medial and, moreover, $\bar{\mathbf{G}}$ is medial too.

Neither mediality nor diagonality is closed under combined products. We only have:

Proposition

Let $\mathbf{G}$ *and* $\mathbf{H}$ *be n-ary hyperalgebras. If* $\overline{\mathbf{G}}$ *and* $\overline{\mathbf{H}}$ *are medial or diagonal, respectively, then so is* $\overline{\mathbf{G} \otimes \mathbf{H}}$.

Lemma

Let $\mathbf{H} = (H, q)$ *and* $\mathbf{G} = (G, p)$ *be n-ary hyperalgebras and let* $f_1, \ldots, f_n \in Hom(\mathbf{H}, \mathbf{G})$*. Let* $f : H \to G$ *be a map such that* $f(x) \in p(f_1(x), \ldots, f_n(x))$ *for every* $x \in H$*. If* $\mathbf{G}$ *is medial, then* f *is a homomorphism from* $\mathbf{H}$ *into* $\mathbf{G}$*.*

Definition

Let $\mathbf{H} = (H, q)$ and $\mathbf{G} = (G, p)$ be *n*-ary hyperalgebras and let $\mathbf{G}$ be medial. The *power* of $\mathbf{G}$ and $\mathbf{H}$ is the *n*-ary hyperalgebra $\mathbf{G}^{\mathbf{H}} = (Hom(\mathbf{H}, \mathbf{G}), r)$ where, for every $f_1, \ldots, f_n \in Hom(\mathbf{H}, \mathbf{G})$, $r(f_1, \ldots, f_n) = \{f \in G^H; f(x) \in p(f_1(x), \ldots, f_n(x))$ for each $x \in H\}$.

Theorem

Let $\mathbf{G}, \mathbf{H}$ *be n-ary hyperalgebras. If* $\mathbf{G}$ *is medial (medial and diagonal), then so is* $\mathbf{G}^{\mathbf{H}}$.

We denote by $MDHYP_n$ the full subconstruct of HYP_n whose objects are the n-ary hyperalgebras that are medial and diagonal.

Lemma

Let $\mathbf{G} \in MDHYP_n$ and $\mathbf{H}, \mathbf{K} \in HYP_n$. Then the canonical bijection $\varphi : (G^H)^K \to G^{H \times K}$ restricted to $Hom(\mathbf{K}, \mathbf{G^H})$ is a bijection of $Hom(\mathbf{K}, \mathbf{G^H})$ onto $Hom(\mathbf{H} \otimes \mathbf{K}, \mathbf{G})$.

Given $\mathbf{G} \in MDHYP_n$ and $\mathbf{H} = (H, q) \in HYP_n$, the *evaluation map* is the map $e : \mathbf{H} \otimes \mathbf{G^H} \to \mathbf{G}$ defined by $e(y, f) = f(y)$ whenever $y \in H$ and $f \in Hom(\mathbf{H}, \mathbf{G})$.

Theorem

Let $\mathbf{G} \in MDHYP_n$ and $\mathbf{H} \in HYP_n$. Then the pair $(\mathbf{G^H}, e)$, where $e : \mathbf{H} \otimes \mathbf{G^H} \to \mathbf{G}$ is the evaluation map, is a co-universal arrow for $\mathbf{G}$ with respect to the functor $\mathbf{H} \otimes - : HYP_n \to HYP_n$.

Remark

For every full subconstruct $\mathcal{A}$ of $MDHYP_n$ closed under both the powers and combined products, and every $\mathcal{A}$-*object* $\mathbf{H}$, the functor $\mathbf{H} \otimes - : \mathcal{A} \to \mathcal{A}$ is a co-adjoint (with the corresponding adjoint being the functor $-^{\mathbf{H}} : \mathcal{A} \to \mathcal{A}$). Consequently, if the functor $\mathbf{H} \otimes - : \mathcal{A} \to \mathcal{A}$ is naturally isomorphic to the functor $\mathbf{H} \times - : \mathcal{A} \to \mathcal{A}$, then $\mathcal{A}$ is cartesian closed. Thus, we get the known result that the full subconstruct of $MDHYP_n$ given by its idempotent objects is cartesian closed.

Corollary

Let $\mathbf{G}, \mathbf{H}, \mathbf{K}$ *be n-ary hyperalgebras. If* $\mathbf{G}$ *is medial and diagonal, then*

$$(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \cong \mathbf{G}^{\mathbf{H} \otimes \mathbf{K}}.$$

Remark

The previous Theorem implies the known result that the first exponential law $(\mathbf{G}^{\mathbf{H}})^{\mathbf{K}} \cong \mathbf{G}^{\mathbf{H} \times \mathbf{K}}$ is valid whenever $\mathbf{G}, \mathbf{H}, \mathbf{K}$ are n-ary hyperalgebras, $\mathbf{G}$ medial and diagonal and $\mathbf{H}, \mathbf{K}$ idempotent. But the Theorem says that, for the combined product, the first exponential law is satisfied for any n-ary hyperalgebras $\mathbf{H}, \mathbf{K}$, not only for the idempotent ones.

Proposition

Let $\mathbf{G}_i, i \in I$, be a non-empty family of medial n-ary hyperalgebras and $\mathbf{H}$ be an n-ary hyperalgebra. If $\bigotimes_{i\in I} \mathbf{G_i}$ is medial, then

$$\bigotimes_{i\in I} \mathbf{G_i^H} \preceq \left(\bigotimes_{\mathbf{i}\in\mathbf{I}} \mathbf{G_i}\right)^{\mathbf{H}}.$$

Remark

It may easily be shown that, in the previous Theorem, we may write $\cong$ instead of $\preceq$ provided that $\mathbf{G_i}$ is idempotent for every $i \in I$. We then obtain the known second exponential law for the direct product $\prod_{i\in I} \mathbf{G_i^H} \cong (\prod_{\mathbf{i}\in\mathbf{I}} \mathbf{G_i})^{\mathbf{H}}$ (in which case we need not assume $\bigotimes_{i\in I} \mathbf{G_i}$ to be medial because it automatically is).