L-E-Algebras

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Our examples are mostly connected to groups.

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The **intersection** $\mu = \bigcap\{\mu_i \mid i \in I\}$ of $\{\mu_i \mid i \in I\}$ is defined by

$$\mu(x) = \bigwedge_{i \in I} \mu_i(x), \quad \text{for every } x \in A.$$

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For $\mu : A \to L$, the collection of cuts

$$\{\mu_p \mid p \in L\}$$

is a closure system. Conversely, for every closure system $\mathcal{F}$ on A, there is a complete lattice L and an $\mu : A \to L$, such that $\mathcal{F}$ is a collection of cuts of μ.

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ρ is **symmetric** if $\rho(x, y) = \rho(y, x)$ for all $x, y \in A$;

ρ is **transitive** if $\rho(x, y) \geqslant \rho(x, z) \wedge \rho(z, y)$ for all $x, y, z \in A$.

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An L-valued equivalence ρ on μ, fulfilling for all $x, y \in A$, $x \neq y$,:

$$\text{if } \rho(x,x) \neq 0, \text{ then } \rho(x,x) > \rho(x,y),$$

is an ***L*-valued equality** relation on μ.

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For any operation f from F with arity greater than 0, $f : A^n \to A, n \in \mathbb{N}$, and for all $a_1, \ldots, a_n \in A$,

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant \mu(f(a_1, \ldots, a_n)),$$

and for a nullary operation (constant) $c \in F$, $\mu(c) = 1$.

Let $\mathcal{A} = (A, F)$ be an algebra. An L-valued relation $\rho : A^2 \to L$ is **compatible** with the operations in F if the following holds: for every n-ary operation $f \in F$ and for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$

$$\bigwedge_{i=1}^{n} \rho(a_i, b_i) \leqslant \rho(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)), \text{ and}$$

$$\rho(c, c) = 1 \text{ for every constant (nullary operation) } c \in F.$$

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An ***L*-valued equality** on μ is an L-valued congruence on μ, such that

$$\rho(x, x) \neq 0 \text{ implies } \rho(x, x) > \rho(x, y).$$

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An L-valued compatible function μ on $\mathcal{A}$ **satisfies identity** $u \approx v$ **with respect to** L**-valued equality** E^μ on μ, if the following condition is fulfilled for all $a_1, \ldots, a_n \in A$ and the term-operations u^A and v^A on $\mathcal{A}$ corresponding to terms u and v respectively:

$$\bigwedge_{i=1}^{n} \mu(a_i) \leqslant E^\mu(u^A(a_1, \ldots, a_n), v^A(a_1, \ldots, a_n)).$$

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Proposition

Let $u \approx v$ be an identity which holds on an algebra $\mathcal{A}$. If $\mu : A \to L$ is an L-valued compatible function on $\mathcal{A}$, and E^{μ} an L-valued equality on μ, then the identity $u \approx v$ is satisfied on μ with respect to E^{μ}.

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be a structure in which $\mathcal{A} = (A, F)$ is an algebra with a set F of operations, $\mu : A \to L$ is an L-valued compatible function on $\mathcal{A}$, $E^{\mu} : A^2 \to L$ is an L-valued equality on μ.

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If, in addition, $\mathcal{F}$ is a collection of identities, and every identity from $\mathcal{F}$ is valid on μ with respect to E^{μ}, then we say that $\bar{\mathcal{A}}$ satisfies all identities from $\mathcal{F}$.

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$$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z,$$

$$x \cdot e \approx x,\ e \cdot x \approx x,$$

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$\mathcal{G} = (G, \cdot, ^{-1}, e)$ is the **underlying algebra** of L-E-group $\bar{\mathcal{G}}$.

According to the definitions, the fact that μ is an L-valued compatible function on $\mathcal{G}$ means that for all $x, y \in G$

- $\mu(x \cdot y) \geqslant \mu(x) \wedge \mu(y)$,
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In addition, the requirement that $\bar{\mathcal{G}}$ fulfills the listed group identities, means that for all x, y, z from G,

(*i*) $E^{\mu}(x \cdot (y \cdot z), (x \cdot y) \cdot z) \geqslant \mu(x) \wedge \mu(y) \wedge \mu(z)$,

(*ii*) $E^{\mu}(x \cdot e, x) \geqslant \mu(x)$ and $E^{\mu}(e \cdot x, x) \geqslant \mu(x)$,

(*iii*) $E^{\mu}(x \cdot x^{-1}, e) \geqslant \mu(x)$ and $E^{\mu}(x^{-1} \cdot x, e) \geqslant \mu(x)$.

In the following we deal only with complete lattices with the property that

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In other words, L is a lattice in which 0 *is not meet-irreducible*.

Theorem

Let $\mathcal{A} = (A, F)$ be an algebra of a type τ, L a complete lattice in which 0 is not meet-irreducible, and $\mu : A \to L$ an L-valued compatible function on $\mathcal{A}$, such that for every $x \in A$, $\mu(x) \neq 0$. Let also Σ be a set of identities of the type τ. Suppose that μ satisfies Σ with respect to an L-valued equality E^{μ}. Then, there is the smallest L-valued equality $E^{\mu}_{\Sigma} : A^2 \to L$ on μ, such that $\bar{\mathcal{A}} = (\mathcal{A}, \mu,\ E^{\mu}_{\Sigma})$ is an L-E-algebra fulfilling Σ.

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If E^{μ}_{Σ} is the greatest L-valued equivalence on μ, i.e., if for all $x, y \in A$

$$E^{\mu}_{\Sigma}(x, y) = \mu(x) \wedge \mu(y),$$

then $\bar{\mathcal{A}}$ is a trivial, one-element algebra.

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If E^{μ}_{Σ} is a diagonal L-valued equality, namely if

$$E^{\mu}_{\Sigma}(x, y) = \begin{cases} \mu(x) & \text{if } x = y \\ 0 & \text{else} \end{cases},$$

then also the algebra $\mathcal{A}$ satisfies Σ.

Theorem

Let $\mathcal{G} = (G, \cdot, ^{-1}, e)$ be an algebra of the type 2,1,0, L a complete lattice, and $\mu : G \to L$ an L-valued compatible function on $\mathcal{G}$. Suppose that μ satisfies identities

$$x \cdot (y \cdot z) \approx (x \cdot y) \cdot z,$$

$$x \cdot e \approx x,\; e \cdot x \approx x,$$

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with respect to an L-valued equality E^{μ}. Then, there is the smallest L-valued equality $E_g^{\mu} : A^2 \to L$ on μ, such that $\bar{\mathcal{G}} = (\mathcal{G}, \mu,\ E_g^{\mu})$ is an L-E-group.

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a	q	r	p	q
b	p	p	q	p
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$\bar{\mathcal{G}} = (\mathcal{G}, \mu,\ E^{\mu})$ is an L-E-group.

Theorem

Let $\mathcal{A} = (A, F)$ be an algebra which satisfies a set of identities Σ, $\mu : A \to L$ an L-valued compatible function on $\mathcal{A}$, and E^{μ} an L-valued equality on μ. Then, $\bar{A} = (\mathcal{A}, \mu, \ E^{\mu})$ is an L-E-algebra which also fulfils Σ.

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Let $\bar{\mathcal{A}} = (\mathcal{A}, \mu,\ E^{\mu})$ *be an L-E-algebra such that for every* $x \in A$, $\mu(x) \neq 0$. *Let also* $t(x)$ *be a term depending on a variable* x *only. Then, identity* $t(x) \approx x$ *holds on* $\bar{\mathcal{A}}$ *if and only if the same identity holds on* $\mathcal{A}$.

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An element $x \in A$ such that $\mu(x) > 0$ is idempotent in an L-E-algebra $\bar{\mathcal{A}}$ where $\mathcal{A}$ has a binary operation $\cdot$, if and only if x is idempotent in $\mathcal{A}$ (i.e., if $x^2 = x$).

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Elements x_0 and y_0 are **solutions** of equations $E^{\mu}(a \cdot x, b)$ and $E^{\mu}(y \cdot a, b)$, respectively. If $\mu(x_0) = 0$ (analogously $\mu(y_0) = 0$), then obviously x_0 (y_0) is a solution of the corresponding equation; we say that it is a **trivial solution**.

Theorem

Let $\bar{\mathcal{G}} = (\mathcal{G}, \mu,\ E^{\mu})$ *be an L-E-group. Then, L-valued equations*

$$(i)\ \ E^{\mu}(a \cdot x, b)\ \ \text{and}\ \ (ii)\ \ E^{\mu}(y \cdot a, b)$$

have nontrivial solutions for arbitrary $a, b \in G$*, such that* $\mu(a) \wedge \mu(b) \neq 0$.

L-E-subalgebra

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Let A be a nonempty set, $\nu : A \to L$ a nonzero L-valued subfunction of an L-valued function $\mu : A \to L$, E^μ an L-valued relation on μ, and $E^\nu : A^2 \to L$ an L-valued relation on ν. We say that E^ν is a **restriction** of E^μ to ν if

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Lemma

Let $\nu : A \to L$ be a nonzero L-valued subfunction of $\mu : A \to L$, and E^μ an L-valued relation on μ. Then a restriction E^ν of E^μ to ν is an L-valued relation on ν.

Proposition

If $E^\mu : A^2 \to L$ is an L-valued equality on $\mu : A \to L$, then the restriction E^ν of E^μ to a nonzero L-valued subfunction ν of μ is an L-valued equivalence on ν. In addition, if μ and ν are L-valued compatible functions on an algebra $\mathcal{A} = (A, F)$, and E^μ is compatible with operations in F, then also E^ν is compatible.

Let $\bar{\mathcal{A}}^\mu = (\mathcal{A}, \mu,\ E^\mu)$ and $\bar{\mathcal{A}}^\nu = (\mathcal{A}, \nu,\ E^\nu)$ be L-E-algebras over the same underlying algebra $A = (A, F)$. We say that $\bar{\mathcal{A}}^\nu$ is an ***L*-*E*-subalgebra** of L-E-algebra $\bar{\mathcal{A}}^\mu$, if ν is an L-valued function on μ and E^ν is a restriction of E^μ to ν.

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Theorem

Let $\bar{\mathcal{A}}^{\mu} = (\mathcal{A}, \mu,\ E^{\mu})$ be an L-E-algebra and $E^1 : A^2 \to L$ an L-valued relation on A, such that $E^1 \leqslant E^{\mu}$. Let E^1 fulfils all L-valued equality properties except reflexivity. In addition, let E^1 satisfies also the following condition:

$$E^1(x, y) = E^{\mu}(x, y) \wedge E^1(x, x) \wedge E^1(y, y).$$

Now, let $\nu : A \to L$ be defined by $\nu(x) := E^1(x, x)$, for every $x \in A$. Then, $\bar{\mathcal{A}}^{\nu} = (\mathcal{A}, \nu,\ E^1)$ is an L-E-subalgebra of L-E-algebra $\bar{\mathcal{A}}^{\mu}$.

Theorem

Let $\bar{\mathcal{G}}^{\mu} = (\mathcal{G}, \mu, E^{\mu})$ be an L-E-group, $\nu : G \to L$ a nonzero L-valued function of μ, and E^{ν} a restriction of E^{μ} to ν. Then the structure $\bar{\mathcal{G}}^{\nu} = (\mathcal{G}, \nu, E^{\nu})$ is an L-E-subgroup of $\bar{\mathcal{G}}^{\mu}$ if and only if it is an L-E-algebra.

Theorem

$\{\bar{\mathcal{G}}^{\mu_i} = (\mathcal{G},\ \mu_i,\ E^{\mu_i}) \mid i \in I\}$ be a nonempty family of L-E-subgroups of an L-E-group $\bar{\mathcal{G}}^{\mu} = (\mathcal{G},\ \mu,\ E^{\mu})$, where $\mathcal{G} = (G, \cdot, ^{-1}, e)$ is a given algebra. Further, for all $x, y \in G$, such that $x \neq y$, and $\bigwedge_{i\in I} \mu_i(x) > 0$, let

$$E^{\mu}(x,y) \wedge \bigwedge_{i\in I} \mu_i(x) \wedge \bigwedge_{i\in I} \mu_i(y) < \bigwedge_{i\in I} \mu_i(x).$$

Finally, let $\delta = \bigcap_{i\in I} \mu_i$ and let E^{δ} be the restriction of E^{μ} to δ. Then the structure $\bar{\mathcal{G}}^{\delta} = (\mathcal{G},\ \delta,\ E^{\delta})$, is an L-E-subgroup of L-E-group $\bar{\mathcal{G}}$.

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Let $\bar{\mathcal{A}}^{\mu} = (\mathcal{A}, \mu, \ E^{\mu})$ *be an* L*-*E*-algebra, and* Σ *a set of identities. Then,* $\bar{\mathcal{A}}^{\mu}$ *fulfils* Σ *if and only if for every* $p \in L$*, the cut* μ_p *is a subalgebra of* $\mathcal{A}$*, the cut relation* E^{μ}_{p} *is a congruence on* μ_p*, and the quotient structure* μ_p / E^{μ}_{p} *satisfies* Σ*.*

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Theorem

Let $\bar{\mathcal{A}}^{\mu} = (\mathcal{A}, \mu,\ E^{\mu})$ be an L-E-algebra fulfiling a set of identities Σ, such that $\mu(x) \neq 0$ for every $x \in A$, and let E^{μ} fulfils the following:

$$\text{for all } x, y \in A \text{ such that } x \neq y,\ E^{\mu}(x, y) < \bigwedge_{z \in A} \mu(z).$$

Then, the underlying algebra $\mathcal{A}$ of $\overline{\mathcal{A}}$ also satisfies Σ.

Theorem

Let $\bar{\mathcal{G}}^{\mu} = (\mathcal{G}, \mu,\ E^{\mu})$ be an L-E-algebra. Then, $\bar{\mathcal{G}}^{\mu}$ is an L-E-group if and only if for every $p \in L$, the cut μ_p is a subalgebra of $\mathcal{G}$, the cut relation E^{μ}_p is a congruence on μ_p, and the quotient structure μ_p/E^{μ}_p is a group.

Theorem

Let $\bar{\mathcal{G}}^\mu = (\mathcal{G}, \mu, \ E^\mu)$ be an L-E-algebra. Then, $\bar{\mathcal{G}}^\mu$ is an L-E-group if and only if for every $p \in L$, the cut μ_p is a subalgebra of $\mathcal{G}$, the cut relation E_p^μ is a congruence on μ_p, and the quotient structure μ_p / E_p^μ is a group.

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Let $\bar{\mathcal{G}}^\mu = (\mathcal{G}, \mu, \ E^\mu)$ be an L-E-group, such that $\mu(x) \neq 0$ for every $x \in G$, and let E^μ fulfils the following:

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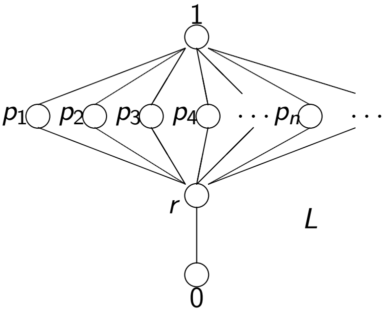

$$\mu := \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \dots & \mathbf{n} & \dots \\ 1 & p_1 & p_2 & p_3 & \dots & p_n & \dots \end{pmatrix}.$$

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E^μ	**0**	**1**	**2**	**3**	**4**	**5**	...
0	1	0	r	0	r	0	$\cdots$
1	0	p_1	0	r	0	r	$\cdots$
2	r	0	p_2	0	r	0	$\cdots$
3	0	r	0	p_3	0	r	$\cdots$
4	r	0	r	0	p_4	0	$\cdots$
5	0	r	0	r	0	p_5	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

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2	r	0	p_2	0	r	0	$\cdots$
3	0	r	0	p_3	0	r	$\cdots$
4	r	0	r	0	p_4	0	$\cdots$
5	0	r	0	r	0	p_5	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

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$\mathbf{0}$	$\mathbf{0}$	$\mathbf{n}$
$\mathbf{n}$	$\mathbf{n}$	$\mathbf{0}$

;

$E^{\mu}_{p_n}$	$\mathbf{0}$	$\mathbf{n}$
$\mathbf{0}$	1	0
$\mathbf{n}$	0	1

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$\oplus$	**0**	**n**
0	**0**	**n**
n	**n**	**0**

;

$E^{\mu}_{p_n}$	**0**	**n**
0	1	0
n	0	1

.

For every $p_n \in L$, the quotient structure $\mu_{p_n}/E^{\mu}_{p_n}$ is a two-element group, isomorphic to μ_{p_n}.

Thank you for your attention !