Tense Operators on Effect Algebras

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Outline

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Introduction - tense operators on the unit interval

For MV-algebras, the so-called tense operators were already introduced by Diaconescu and Georgescu. Tense operators express the quantifiers “it is always going to be the case that” and “it has always been the case that” and hence enable us to express the dimension of time in the logic.

A crucial problem concerning tense operators is their representation. Having a MV-algebra with tense operators, Diaconescu and Georgescu asked if there exists a frame such that each of these operators can be obtained by their construction for $[0,1]$. We solved this problem with M. Botur for semisimple MV-algebras, i.e. those having a full set of MV-morphisms into a standard MV-algebra $[0,1]$. The aim of this lecture is to extend these results to the setting of effect algebras.

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Tense operators on $[0, 1]$

Tense operators were used to express the dimension of time into Łukasiewicz logics.

- Let T be a time scale,
- then elements $f(t)$ from $[0, 1]^T$ correspond to the evaluation of the validity of the formula f in time.

For a moment, let T be the interval $[0, 2.5]$.

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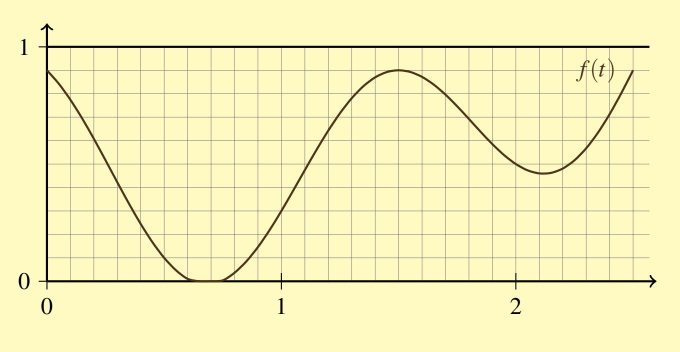

Tense operators on $[0, 1]$

On the time scale T we will introduce a relation i $\rho \subseteq T^2$.

Moreover, we introduce operators G a H on $[0, 1]^T$ as follows:

- $x\rho y$ means that the moment x is before the moment y.
- Gf means that f will be true in future with at least the same degree as f is now.
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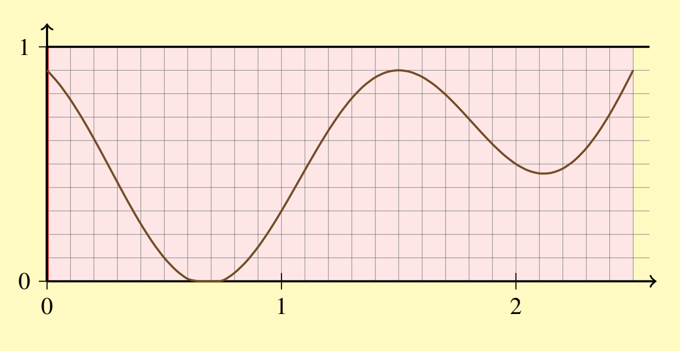

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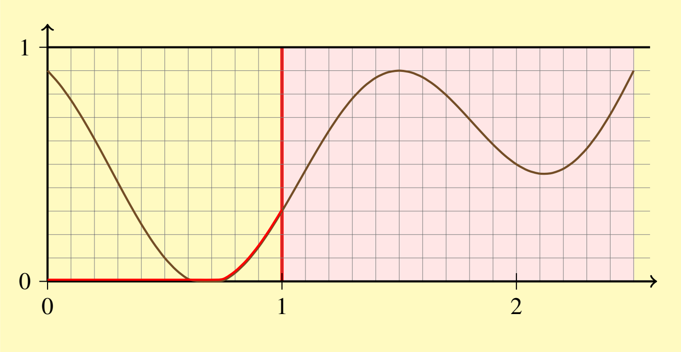

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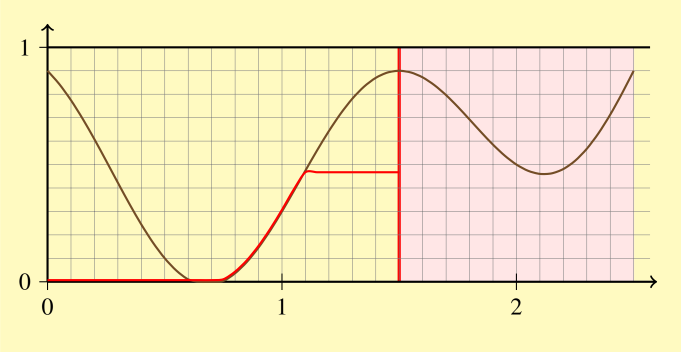

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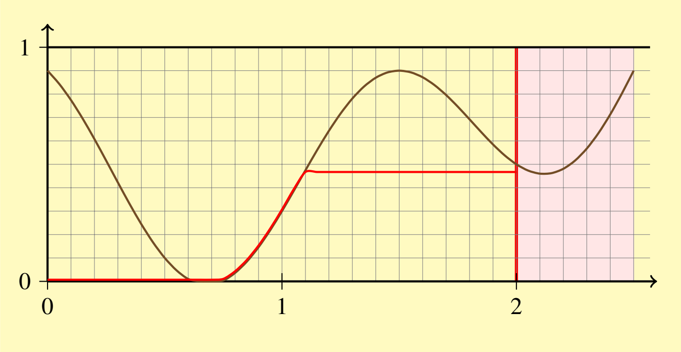

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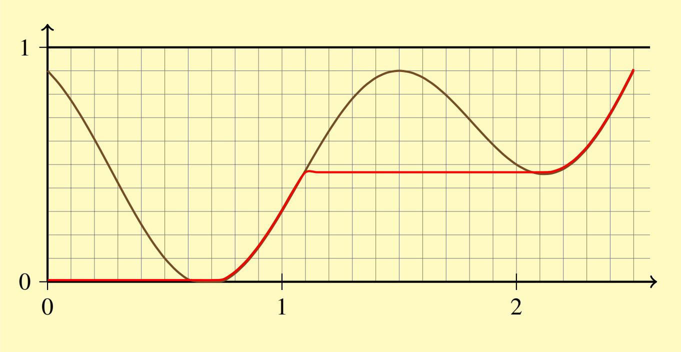

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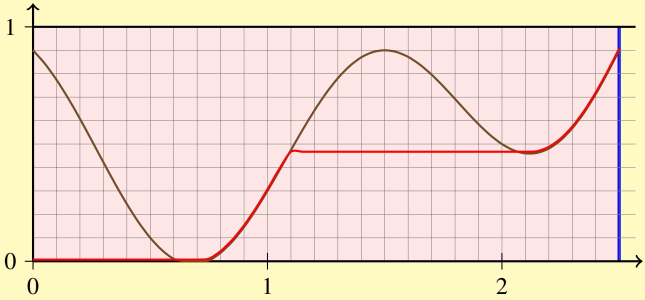

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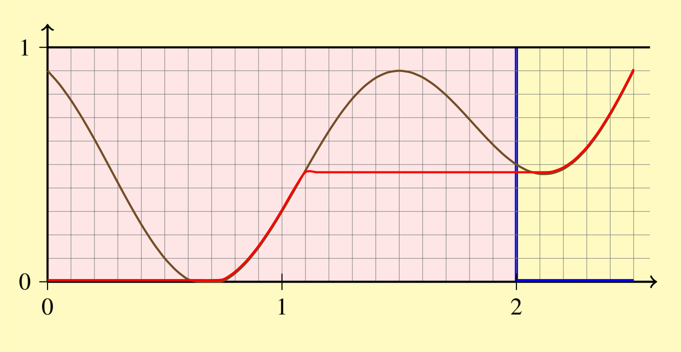

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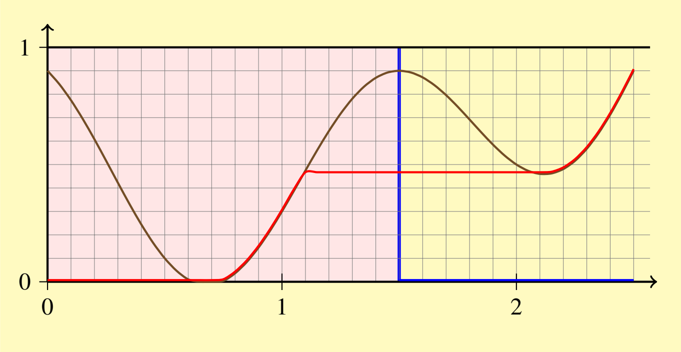

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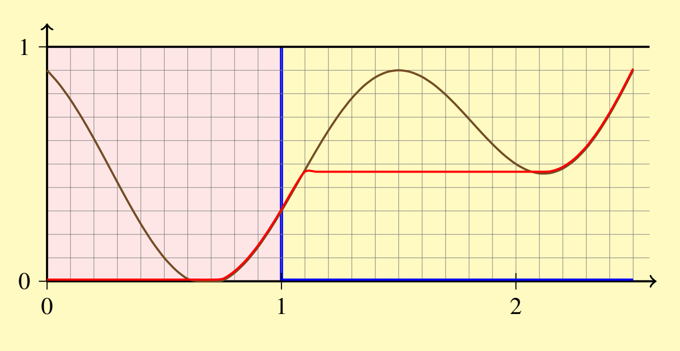

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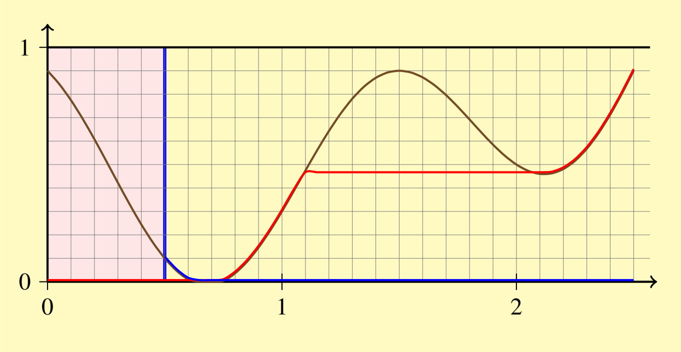

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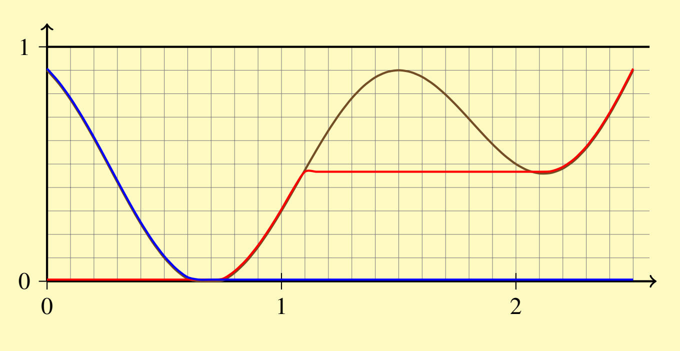

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Basic definition – MV-algebras

Definition (Chang, 1958)

An ***MV-algebra*** $\mathscr{M} = (M; \oplus, \odot, \neg, 0, 1)$ is a structure where $\oplus$ is associative and commutative with neutral element 0, and, in addition, $\neg 0 = 1, \neg 1 = 0, x \oplus 1 = 1, x \odot y = \neg(\neg x \oplus \neg y)$, and $y \oplus \neg(y \oplus \neg x) = x \oplus \neg(x \oplus \neg y)$ for all $x, y \in M$.

MV-algebras are a natural generalization of Boolean algebras. Namely, whilst Boolean algebras are algebraic semantics of Boolean two-valued logic, MV-algebras are algebraic semantics for Łukasiewicz many valued logic.

Example

An example of a MV-algebra is the real unit interval $[0,1]$ equipped with the operations

$$\neg x = 1 - x, x \oplus y = \min(1, x + y), x \odot y = \max(0, x + y - 1)$$

We refer to it as a *standard MV-algebra*.

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Basic definitions – MV-algebras

Every MV-algebra $\mathscr{M}$ determines a dual MV-algebra $\mathscr{M}^{op} = (M; \oplus^{op}, \odot^{op}, \neg^{op}, 0^{op}, 1^{op})$ such that $\oplus^{op} = \odot, \odot^{op} = \oplus$, $\neg^{op} = \neg, 0^{op} = 1$ and $1^{op} = 0$.

On every MV-algebra $\mathscr{M}$, a partial order $\leq$ is defined by the rule

$$x \leq y \Longleftrightarrow \neg x \oplus y = 1.$$

In this partial order, every MV-algebra is a distributive lattice bounded by 0 and 1.

An MV-algebra is said to be *linearly ordered* (or a *MV-chain*) if the order is linear.

A frame (T, ρ) is a pair where T is a non-empty set and ρ is a relation on T.

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Basic results – frames on linearly ordered complete MV-algebras

Theorem (D. Diaconescu and G. Georgescu)

Let $\mathscr{M}$ *be a linearly ordered complete MV-algebra,* (T,ρ) *be a frame,* G *and* H *be maps from* M^T *into* M^T *defined by*

$$\begin{array}{rcl} G(p)(s) & = & \bigwedge\{p(t) \mid t \in T, s\rho t\}, \\ H(p)(s) & = & \bigwedge\{p(t) \mid t \in T, t\rho s\} \end{array}$$

for all $p \in M^T$ *and* $s \in T$*. Then*

(T1) $G(1) = H(1) = 1$,

(T2) $G(x) \odot G(y) \leq G(x \odot y)$, $H(x) \odot H(y) \leq H(x \odot y)$,

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(T2) $G(x) \odot G(y) \leq G(x \odot y)$, $H(x) \odot H(y) \leq H(x \odot y)$,

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Hence they are tense operators *in the sense of D. Diaconescu and G. Georgescu.*

Basic results – frames on linearly ordered complete MV-algebras

Theorem (D. Diaconescu and G. Georgescu)

Let $\mathscr{M}$ be a linearly ordered complete MV-algebra, (T,ρ) be a frame, G and H be maps from M^T into M^T defined by

$$
\begin{aligned}
G(p)(s) &= \bigwedge\{p(t) \mid t \in T, s\rho t\}, \\
H(p)(s) &= \bigwedge\{p(t) \mid t \in T, t\rho s\}
\end{aligned}
$$

for all $p \in M^T$ and $s \in T$. Then

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Morphisms of MV-algebras (shortly *MV-morphisms*) are defined as usual, they are functions which preserve the binary operations $\oplus$ and $\odot$, the unary operation $\neg$ and nullary operations 0 and 1.

Definition

Let P, Q be bounded posets and let T be a set of order-preserving maps from P to Q. Then T is called *order reflecting* if

$$((\forall s \in T)\, s(a) \leq s(b)) \implies a \leq b$$

for any elements $a, b \in P$.

An MV-algebra $\mathscr{M}$ is called *semisimple* if it has an order reflecting set T of MV-morphisms from $\mathscr{M}$ to $[0,1]$. We then have an order reflecting MV-morphism $i^T_{\mathscr{M}} : \mathscr{M} \to [0,1]^T$ given as follows:

$$i^T_{\mathscr{M}}(x)(s) = s(x)$$

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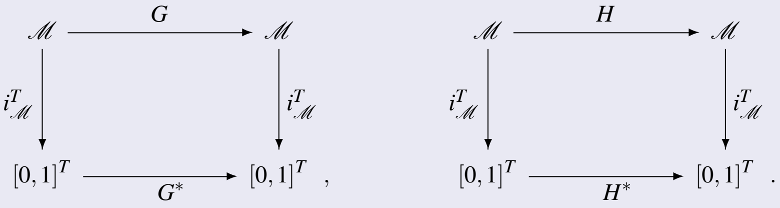
Basic results – The representation theorem for semisimple MV-algebras

Theorem (M. Botur and J.P.)

Let $\mathcal{M}$ be a semisimple MV-algebra with an order reflecting set T of MV-morphisms from $\mathcal{M}$ to $[0,1]$ with tense operators G and H. Then $(\mathcal{M}, G, H)$ can be embedded into the tense MV-algebra $([0,1]^T, G^*, H^*)$ induced by the frame (T, ρ_G), where the relation ρ_G is defined by

$$s\rho_G t \text{ if and only if } s(G(x)) \leq t(x) \text{ for any } x \in M,$$

i.e., the following two diagrams commute:



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(E3) for each $x \in E$ there exists a unique $x' \in E$ such that $x+x' = 1$; x' is called a *supplement* of x

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Having an effect algebra $\mathscr{E} = (E;+,0,1)$, we can introduce the *induced order* $\leq$ on E and the partial operatin $-$ as follows

$$x \leq y \quad \text{if for some } z \in E \quad x+z = y, \text{and in this case } z = y-x.$$

Then $(E;\leq)$ is an ordered set and $0 \leq x \leq 1$ for each $x \in E$.

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Basic definitions – effect algebras and their duals

It is worth noticing that $a + b$ exists in an effect algebra $\mathscr{E}$ if and only if $a \leq b'$ (or equivalently, $b \leq a'$). This condition is usually expressed by the notation $a \perp b$ (we say that a, b are orthogonal).

Dually, we have a partial operation $\cdot$ on E such that $a \cdot b$ exists in an effect algebra $\mathscr{E}$ if and only if $a' \leq b$ in which case $a \cdot b = (a' + b')'$.

This allows us to equip E with a dual effect algebraic operation $\cdot$ such that $\mathscr{E}^{op} = (E; \cdot, 1, 0)$ is again an effect algebra, $'^{\mathscr{E}^{op}} = '^{\mathscr{E}} = '$ and $\leq_{\mathscr{E}^{op}} = \leq^{op}$.

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Prototypical example for effect algebras

Example

The most important example of an effect algebra for quantum mechanical investigations is a Hilbert space effect algebra. Let $\mathscr{H}$ be a complex Hilbert space and let $\mathscr{E}(\mathscr{H})$ be the set of linear operators on $\mathscr{H}$ that satisfy $0 \leq A \leq I$.

That is $0 \leq \langle Ax, x \rangle \leq \langle Ix, x \rangle$ for all $x \in \mathscr{H}$. For $A, B \in \mathscr{E}(\mathscr{H})$ we write $A \perp B$ if $A + B \in \mathscr{E}(\mathscr{H})$ and in this case we define $A +_{eff} B = A + B$. If we define $A' = I - A$ for $A \in \mathscr{E}(\mathscr{H})$, it is clear that $(\mathscr{E}(\mathscr{H}); +_{eff}, 0, I)$ is an effect algebra.

Denoting the set of projections on $\mathscr{H}$ by $\mathscr{P}(\mathscr{H})$ we have $\mathscr{P}(\mathscr{H}) \subseteq \mathscr{E}(\mathscr{H})$ and it is clear that $\mathscr{P}(\mathscr{H})$ is an orthoalgebra in $\mathscr{E}(\mathscr{H})$.

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(Q2) $d(0) = 0 = q(0)$,

(Q3) d is order-preserving,

(Q4) $x' \leq x$ implies $x \cdot x = d(x)$,

(Q5) $z \leq x, z \leq y$ and $x' \leq y$ imply $d(z) \leq x \cdot y$,

(Q6) $d(x) \leq x$.

We then say that $\mathscr{E} = (E;+,q,d,0,1)$ is a *q-effect algebra*.

Note that a dual of $\mathscr{E} = (E;+,q,d,0,1)$ is a q-effect algebra $\mathscr{E}^{op} = (E;\cdot,d,q,1,0)$.

Basic notions – morphisms of effect algebras and q-effect algebras

A *morphism of effect algebras* (*morphism of q-effect algebras*) is a map between them such that it preserves the partial operation $+$, (and the unary operations q and d),the bottom and the top elements. In particular, $' : \mathscr{E} \to \mathscr{E}^{op}$ is a morphism of effect algebras (morphism of q-effect algebras).

A map $s : E \to [0,1]$ is called a *state* on $\mathscr{E}$ if $s(0) = 0$, $s(1) = 1$, and $s(x+y) = s(x) + s(y)$ whenever $x+y$ exists in $\mathscr{E}$.

A state $s : E \to [0,1]$ is called an *q-state* on $\mathscr{E}$ if

- $s(q(x)) = s(x) + \min(s(x'), s(x)) = s(x) \oplus s(x)$,
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Examples – effect algebras and q-effect algebras

Example

1. Let $\mathscr{M} = (M; \oplus, \odot, \neg, 0, 1)$ be an MV-algebra. Let us put $x + y := x \oplus y$ whenever $x \leq \neg y$. Then $(M; +, 0, 1)$ is an effect algebra which is a lattice (lattice effect algebra).
2. Let $\mathscr{E} = (E; +, 0, 1)$ be a lattice effect algebra. Let us put

$$d(x) = x \cdot (x \vee x'), \; q(x) = x + (x \wedge x').$$

Then (Q1)-(Q6) are satisfied, i.e., $(E; +, q, d, 0, 1)$ is a q-effect algebra.

For an MV-algebra $\mathscr{M} = (M; \oplus, \odot, \neg, 0, 1)$ then $d(x) = x \odot x$ and $q(x) = x \oplus x$. In what follows we will write $\mathscr{E} = (E; +, \oplus, \odot, 0, 1)$ for a q-effect algebra.

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Outline

Frames on linearly ordered complete MV-algebras

Theorem

Let $\mathscr{M}$ be a linearly ordered complete MV-algebra, (T,ρ) be a frame, G and H be maps from M^T into M^T defined by

$$\begin{aligned} G(p)(s) &= \bigwedge\{p(t) \mid t \in T, s\rho t\}, \\ H(p)(s) &= \bigwedge\{p(t) \mid t \in T, t\rho s\} \end{aligned}$$

for all $p \in M^T$ and $s \in T$. Then

(ET1) $x \leq y$ *implies* $G(x) \leq G(y)$ *and* $H(x) \leq H(y)$,

(ET2) $('\circ G \circ ') \circ H(x) \leq x$, $x \leq H \circ (' \circ G \circ ')(x)$,

(ET3) $G(x) \oplus G(x) = G(x \oplus x)$, $H(x) \oplus H(x) = H(x \oplus x)$,

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q-tense q-effect algebras

Definition

Let $\mathscr{E} = (E; +, \oplus, \odot, 0, 1)$ be a q-effect algebra and let G and H be operators on E satisfying (ET1)-(ET4) from the preceding theorem.

We say that G and H are *q-tense operators* and that $(E; +, \oplus, \odot, 0, 1, G, H)$ is a *q-tense q-effect algebra*.

Jauch-Piron q-states

Definition

Let $\mathscr{E} = (E; +, \oplus, \odot, 0, 1)$ be a q-effect algebra and let $s : E \to [0,1]$ be a q-state. s is called a *Jauch-Piron q-state* if, for all $x, y \in E$, $s(x) = 1 = s(y)$ implies that there is $z \in E$ such that $s(z) = 1$.

If $\mathscr{E}$ is an MV-algebra and s is an MV-algebra morphism from $\mathscr{E}$ to $[0,1]$ then s is a q-state and we always have $s(x \vee y) + s(x \wedge y) = s(x) + s(y)$ for all $x, y \in E$. Hence s is a Jauch-Piron q-state.

If any q-state is Jauch-Piron then $\mathscr{E}$ is called a *q-Jauch-Piron q-effect algebra*.

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Theorem

Let $\mathscr{E}$ be a q-Jauch-Piron q-effect algebra with an order reflecting set T of all q-states and with tense q-operators G and H. Then $(\mathscr{E}, G, H)$ can be embedded into the tense MV-algebra $([0,1]^T, G^*, H^*)$ induced by the frame (T, ρ_G), where the relation ρ_G is defined by

$$s \rho_G t \text{ if and only if } s(G(x)) \leq t(x) \text{ for any } x \in E.$$

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1. We have to show that $s \circ G = \bigwedge\{t \in T \mid t \geq s \circ G\}$.

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3. Put $U = \{z \in E | s \circ G(z) = 1\}$.
4. We can find a prime filter W in $[0,1]^T$, $W \supseteq U$ such that $[x]_W \neq [1]_W$ and $[U]_W = [1]_W$.
5. This yields an MV-morphism $\bar{s} : ([0,1]^T)/W \to [0,1]$ such that $\bar{s}([x]_W) < r$.
6. Put $t(y) = \bar{s}([y]_W)$, $y \in E$. Then $s \circ G \leq t$, $t(x) < r$, a contradiction.

Main representation theorem

Theorem

Let $\mathscr{E}$ be a q-Jauch-Piron q-effect algebra with an order reflecting set T of all q-states and with tense q-operators G and H. Then $(\mathscr{E}, G, H)$ can be embedded into the tense MV-algebra $([0,1]^T, G^*, H^*)$ induced by the frame (T, ρ_G), where the relation ρ_G is defined by

$$s\rho_G t \text{ if and only if } s(G(x)) \leq t(x) \text{ for any } x \in E.$$

The sketch of the proof:

$$\mathscr{E} \xrightarrow{\quad i^T_{\mathscr{E}} \quad} [0,1]^T \xrightarrow{\quad [\ ]_W \quad} ([0,1]^T)/W \xrightarrow{\quad \bar{s} \quad} [0,1]$$

$$U \qquad U \subseteq W, x \notin W \qquad [x]_W \neq [1]_W, [U]_W = \{[1]_W\} \qquad \bar{s}([x]_W) < r.$$

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1. *Using e.g. functional calculus can we naturally introduce* $A \oplus A$ *and* $A \odot A$*, i.e., can we verify that* $(\mathscr{E}(\mathscr{H}); +_{eff}, \oplus, \odot, 0, I)$ *is a q-effect algebra?*
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References

- M. Botur, J. Paseka, On tense MV-algebras, preprint, http://arxiv.org/pdf/1305.3406v1.pdf, (2013).
- J. Burges, Basic tense logic, in: Handbook of Philosophical Logic, vol. II (D. M. Gabbay, F. Günther, eds.), D. Reidel Publ. Comp., 1984, pp. 89–139.
- C.C. Chang, Algebraic analysis of many-valued logics, Trans. Amer. Math. Soc. 88 (1958) 467–490.
- R. Cignoli, I. D'Ottaviano, D. Mundici, Algebraic Foundations of Many-valued Reasoning, Trends in Logic Vol 7, Kluwer Academic Publishers, 2000.
- D. Diaconescu, G. Georgescu, Tense Operators on MV-Algebras and Łukasiewicz-Moisil Algebras, Fundamenta Informaticae 81 (2007) 379–408.

References

- M. Botur, J. Paseka, On tense MV-algebras, preprint, http://arxiv.org/pdf/1305.3406v1.pdf, (2013).
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References

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- C.C. Chang, Algebraic analysis of many-valued logics, Trans. Amer. Math. Soc. 88 (1958) 467–490.
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References

- J. Łukasiewicz, *On three-valued logic*, in L. Borkowski (ed.), Selected works by Jan Łukasiewicz, North-Holland, Amsterdam, 1970, pp. 87-88.
- P. Ostermann, Many-valued modal propositional calculi. Z. Math. Logik Grundlag. Math., 34 (1988) 343–354.
- B. Teheux, A Duality for the Algebras of a Łukasiewicz $n+1$-valued Modal System, Studia Logica 87 (2007) 13–36, doi: 10.1007/s11225-007-9074-5.
- B. Teheux, Algebraic approach to modal extensions of Łukasiewicz logics, Doctoral thesis, Université de Liege, 2009, http://orbi.ulg.ac.be/handle/2268/10887.

References

- J. Łukasiewicz, *On three-valued logic*, in L. Borkowski (ed.), Selected works by Jan Łukasiewicz, North-Holland, Amsterdam, 1970, pp. 87-88.
- P. Ostermann, Many-valued modal propositional calculi. Z. Math. Logik Grundlag. Math., 34 (1988) 343–354.
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- J. Łukasiewicz, *On three-valued logic*, in L. Borkowski (ed.), Selected works by Jan Łukasiewicz, North-Holland, Amsterdam, 1970, pp. 87-88.
- P. Ostermann, Many-valued modal propositional calculi. Z. Math. Logik Grundlag. Math., 34 (1988) 343–354.
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Thank you for your attention.