Strong endomorphism kernel property for modular *p*-algebras and unbounded distributive lattices

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Outline

1. Introduction
2. Modular (and distributive) *p*-algebras
3. Direct product
4. Full characterization
5. End of a SSAOS 2014
6. Bibliography

We shall study the notion of strong endomorphism property defined by Blyth and Silva in 2004. Let A be a universal algebra, $f : A \to A$ be an endomorphism, $\Theta \in Con(A)$ be a congruence on A. f is *compatible* with Θ if $a \equiv b(\Theta) \Rightarrow f(a) \equiv f(b)(\Theta)$.

Endomorphism f is *strong* (*congruence preserving*) (on A), if it is compatible with every congruence $\Theta \in Con(A)$.

Definition

An algebra A has the *strong endomorphism kernel property* (=SEKP) if and only if every congruence relation on A different from the universal congruence ι_A is the kernel of a strong endomorphism on A.

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Blyth and Silva considered the case of Ockham algebras and in particular of MS-algebras. They proved e.g. that

- a finite Boolean algebra has SEKP if and only if it is 2 element BA,
- a finite bounded distributive lattice possesses SEKP if and only if it is a 2 element chain and
- they provided full characterization of MS-algebras having SEKP.

Blyth, J. Fang and Wang in 2013 proved a full characterization of finite distributive double p-algebras and finite double Stone algebras having SEKP.

SEKP for distributive p-algebras and Stone algebras has been also studied and fully characterized by G. Fang and J. Fang in 2013.

J. Fang and Sun fully characterized semilattices with SEKP in 2013.

The main approach in papers most of mentioned papers is done by regarding algebras in question as Ockham algebras and using the duality theory of H. Priestley.

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There is one important universal assumption in the original paper of Blyth and Silva, namely all algebras considered in this paper must contain two nullary operations (denoted by 0 and 1, $0 \neq 1$). This assumption is necessary to prove all important statements in their paper and therefore it seems to be impossible to directly adapt their methods to algebras which do not satisfy this assumption (e.g. $\{1\}$-lattices or unbounded lattices) Let us mention three of these results:

Theorem

If an algebra A has SEKP, then it has at most one maximal congruence.

Corollary

A finite algebra that has SEKP is directly indecomposable.

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A semisimple algebra has SEKP if and only if it is simple.

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As it is easy to check, $\{0,1\}^2$ considered as 4 element distributive lattice with a top element ($\{1\}$-lattice) has SEKP and none of these statements are true for this algebra.

Last theorem can be used also to verify that the only Boolean algebra which possesses SEKP is 2 element one, because every Boolean algebra is semisimple and the only simple Boolean algebra is 2 element Boolean algebra.

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$S-$*algebra* is a p-algebra satisfying the Stone identity $x^* \vee x^{**} = 1$. The $S-$algebra L is a *Stone algebra*, if it is a distributive $S-$algebra.

Katriňák and Mederly shown that every modular p-algebra possesses two important parts:

- the Boolean algebra of *closed* elements $(B(L); +, \wedge, ^*, 0, 1)$, where $x \in B(L)$ if and only if $x = x^{**}$ and

$$x^{**} + y^{**} = (x \vee y)^{**}$$

  for every $x, y \in L$;
- the second key subset of L is the filter $D(L)$ of *dense* elements, x is dense if and only if $x^* = 0$.

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They proved that a modular p-algebra L is uniquely determined (up to isomorphism) by its *associated triple* $(B(L), D(L), \varphi(L))$ where $\varphi(L)$ is a so called structural homomorphism.

Even if the proof needs this triple construction, the first important result can be formulated very simply as

Theorem

Let L be a distributive (modular) p-algebra. Then L satisfies the SEKP if and only if

(i) $B(L) \cong \mathbf{2}$

(ii) $D(L)$ *has SEKP as* $\{1\}$*-lattice.*

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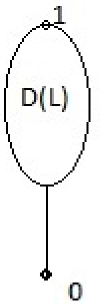

That means that full characterization of Stone algebras with SEKP provide also full characterization of $\{1\}$-lattices with SEKP.

Fortunately, the full characterization of SEKP for Stone lattices was given by Blyth and Silva in 2004 and by Fang and Fang in 2013. Blyth and Silva provided full characterization of (fixed point free, among the others) MS algebras and that covers Stone algebras. Fang and Fang more-less copied their result.

As the Corollary we have the following:

Theorem

A distributive lattice L has SEKP as a $\{1\}$-lattice iff and only if it is

- *1 element lattice, or*
- *$L \cong \{0, 1\}^n$ (as $\{1\}$-lattice) in nontrivial finite case, or*
- *L is isomorphic to a lattice of all cofinite subsets of some infinite set Z (as a $\{1\}$-lattice). This lattice does not have bottom element.*

From this and the result of Blyth and Silva it follows that a bounded distributive lattice has SEKP iff it is a 2 element chain (their theorem states it for a finite case).

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Unlike the things mentioned for algebras with 2 nullary operations, we can use direct product construction e.g. for ({0},{1}, unbounded) lattices as it is provided by the following

Theorem

Let V be a variety with factorable congruences in which every algebra A has one element subalgebra. Let $A_1, A_2, \ldots, A_n \in V$. Then $A_1 \times A_2, \ldots \times A_n$ has SEKP if and only if every A_i, $i = 1, \ldots, n$ has SEKP.

It is known, that any strong endomorphism f of a distributive lattice is a retraction and the $\mathrm{Im} f$ is a convex sublattice of L.
Using this it is easy to see that 4 element chain C_4 does not enjoy SEKP as an unbounded lattice, on the other side, one can easily see that C_3 considered as an unbounded lattice has SEKP (but neither as a $\{1\}$-lattice nor as a $\{0\}$-lattice).

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Corollary

Let a lattice L be a finite product of chains. Then L has SEKP as an unbounded lattice if and only if L is isomorphic to $\mathbf{2}^n \times \mathbf{C_3}^m$ *for some* $m, n = 0, 1, 2, \ldots$

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Mentioned result concerning direct products can be generalized to some special case of so called weak direct product (we are not going into details), but as a consequence e.g.

$$L_1 = \{(a_1, \ldots, a_n, \ldots); a_i = 1 \text{ for all but a finitely many } i\} \subseteq \{0, 1\}^\infty$$

has SEKP as $\{1\}$-lattice, and, e.g.

$$L_2 = \{(a_1, \ldots, a_n, \ldots); a_i = a \text{ for all but a finitely many } i\} \subseteq \{0, a, 1\}^\infty$$

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First example, i.e. L_1 is nothing more than a lattice of all cofinite subsets of an infinite set.

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2 Modular (and distributive) p-algebras

3 Direct product

4 Full characterization

5 End of a SSAOS 2014

6 Bibliography

Finite case

Theorem

Let L be unbounded finite distributive lattice. TFAE

- *L has SEKP*
- *L does not have C_4 as a retract*
- *L does not have C_4 as a homomorphic image*
- *poset $P(L)$ of proper prime ideals (and/or $Ji(L)$ of all join irreducible elements) has height (length) 1*
- *there exists $c \in L$ such that for every $x \in (c\uparrow \cup c\downarrow) \setminus \{c\}$ $[x, c]$ (if $x < c$) and $[c, x]$ (if $x > c$) are Boolean*

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Theorem

Let L be unbounded (infinite) distributive lattice. TFAE

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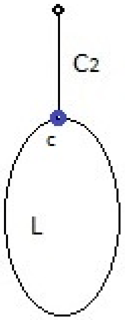
- *L is locally finite and there exists* $c \in L$ *such that for every* $x \in (c\uparrow \cup c\downarrow) \setminus \{c\}$ $[x, c]$ *(if* $x < c$*) and* $[c, x]$ *(if* $x > c$*) are (finite) Boolean*

One can prove that if we have a distributive lattice of the form



than it has SEKP iff C_2 has SEKP as $\{0\}$-lattice (this is true) and L has SEKP as $\{1\}$-lattice. The only possible choice for c from previous 2 theorems is indicated on the picture and we get another proof of the characterization for $\{1\}$-lattices as a consequence.

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Thank you for the attention.

I would like to thank the organizers for organizing this beautiful conference.

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