Sheaf spaces of partially ordered quasigroups

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Quasigroups

A **quasigroup** is an algebra $(Q, \cdot, \backslash, /)$ with three binary operations satisfying the following identities:

$$y\backslash(y \cdot x) \approx x; \qquad (x \cdot y)/y \approx x$$
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Conversely, if Q is a groupoid such that the equations $b \cdot x = a$ and $y \cdot b = a$ have unique solutions $x, y \in Q$, then Q is a quasigroup, where $b\backslash a$ and a/b are defined as the solution of the equation $b \cdot x = a$ or $x \cdot b = a$, respectively.

Partially ordered quasigroups

A quasigroup $(Q, \cdot, \backslash, /)$ with a binary relation $\leq$ is called a **partially ordered quasigroup** (po-quasigroup) if

(i) $(Q, \leq)$ is a partially ordered set,

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$(R, \cdot, \backslash, /, \leq)$, where $\leq$ is a natural linear order on R (real numbers) and $x \cdot y = 2x + y + 3$, $y \backslash x = x - 2y - 3$, $x/y = \frac{x-y-3}{2}$.

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Analogously to the case of the lattice ordered groups it can be proved that for l-quasigroups the following identities hold

- $a(b \vee c) = ab \vee ac;$ $\quad (b \vee c)a = ba \vee ca,$
  $a(b \wedge c) = ab \wedge ac;$ $\quad (b \wedge c)a = ba \wedge ca.$
- $(b \vee c)/a = (b/a) \vee (c/a);$ $\quad a\backslash(b \vee c) = (a\backslash b) \vee (a\backslash c),$
  $(b \wedge c)/a = (b/a) \wedge (c/a);$ $\quad a\backslash(b \wedge c) = (a\backslash b) \wedge (a\backslash c).$
- $a/(b \vee c) = (a/b) \wedge (a/c);$ $\quad (b \vee c)\backslash a = (b\backslash a) \wedge (c\backslash a),$
  $a/(b \wedge c) = (a/b) \vee (a/c);$ $\quad (b \wedge c)\backslash a = (b\backslash a) \vee (c\backslash a).$

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We say that θ is **a convex congruence relation** on partially ordered quasigroup $\mathcal{Q}$ if there exists $a \in Q$ such that $[a]\theta$ is a convex subset of Q.

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We say that θ is **a convex congruence relation** on partially ordered quasigroup $\mathcal{Q}$ if there exists $a \in Q$ such that $[a]\theta$ is a convex subset of Q.

We say that θ is a **directed congruence relation** on $\mathcal{Q}$ if there exists $a \in Q$ such that the congruence class $[a]\theta$ is a directed subset of $\mathcal{Q}$.

Partially ordered quasigroups

Put

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$$[x]\theta \leq [y]\theta \text{ iff there exist } a \in [x]\theta, b \in [y]\theta \text{ such that } a \leq b$$

Let $(Q, \cdot, \backslash, /, \leq)$ be a po-quasigroup. A quotient-quasigroup $(Q, \cdot, \backslash, /)/\theta$ with the relation $\leq$ defined above is a partially ordered quasigroup if and only if θ is a convex congruence relation on $\mathcal{Q}$.

Partially ordered quasigroups

If $\mathcal{Q}$ be a lattice ordered quasigroup, then a congruence relation θ on $\mathcal{Q}$ is directed and convex if and only if it has the following Substitution Property

$$x\theta y \Rightarrow (x \vee z)\ \theta\ (y \vee z) \text{ and } (x \wedge z)\ \theta\ (y \wedge z) \text{ for each } z \in Q$$

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Let $\mathbf{Q}$ be a lattice ordered quasigroup, θ be a convex directed congruence relation on $\mathcal{Q}$. Then $\mathcal{Q}/\theta$ is a lattice ordered quasigroup.

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The convex directed congruence relations of a Riesz quasigroup $\mathcal{Q}$ form a distributive sublattice in the lattice of all congruence relations of $\mathcal{Q}$.

Sheaf spaces of po-quasigroups

Let E and X be topological spaces. Let $\sigma : E \to X$ be a local homeomorphism (i.e., each point $s \in E$ has a neighborhood V such that $\sigma(V)$ is an open set in X and the restricted mapping $\sigma|_V : V \to \sigma(V)$ is a homeomorphism).

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For $x \in X$, the set $E_x = \sigma^{-1}(x)$ is called the **fibre** over x.

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Let U be an open set in X.

A continuous mapping $f : U \to E$ such that $f(x) \in \sigma^{-1}(x)$ for all $x \in U$ is called a **continuous local section** of σ over U.

If σ is surjective and $U = X$, f is called a **continuous global section**.

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By $E \Delta E$ we denote the set $\bigcup_{x \in X}(E_x \times E_x)$ with the induced topology from $E \times E$.

Sheaf spaces of po-quasigroups

Let E and X be topological spaces and let $\sigma : E \to X$ be a surjective local homeomorphism. We say that a triplet (E, X, σ) is a **sheaf space of po-quasigroups** on the space X if

(i) each fibre E_x is a po-quasigroup,

(ii) the mappings $(s, t) \mapsto s \cdot t$, $(s, t) \mapsto t \backslash s$ and $(s, t) \mapsto s/t$ from $E \Delta E$ to E are continuous.

A sheaf space of po-quasigroups (E, X, σ) is said to be a **sheaf space of l-quasigroups** if each fibre E_x is an l-quasigroup and the mappings $(s, t) \mapsto s \vee t$, $(s, t) \mapsto s \wedge t$ from $E \Delta E$ to E are continuous.

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Example
Consider a topological space X, and an po-quasigroup $\mathcal{Q}$ with the discrete topology. Then $X \times Q$, endowed with product topology, with projection map $\sigma(x, q)$ = x is a sheaf space of po-quasigroups.

Sheaf spaces of po-quasigroups

Let (E, X, σ) be a sheaf space of po-quasigroups. Denote by $\mathcal{R}$ the set of all continuous global sections of σ and define the relation $\leq$ on $\mathcal{R}$ by

$$g \leq h \Leftrightarrow g(x) \leq h(x) \text{ for all } x \in X.$$

If $\mathcal{R} \neq \emptyset$, then $\mathcal{R}$ with the operations $\cdot$, $/$, $\backslash$ defined componentwise and the relation $\leq$ defined above is a po-subquasigroup of the direct product $\prod_{x \in X} E_x$.

Theorem

Let $\mathcal{Q}$ be a po-quasigroup and let X be a topological space. Suppose that for each $x \in X$ there exists a convex congruence relation θ_x on $\mathcal{Q}$ such that the following conditions are satisfied

(i) for all $g, h \in Q$, the set $U_{gh} = \{x \in X \mid [g]\theta_x = [h]\theta_x\}$ is open in X,

(ii) if $[g]\theta_x \leq [h]\theta_x$ for each $x \in X$, then $g \leq h$.

Then $\mathcal{Q}$ can be o-embedded into a po-quasigroup of the continuous global sections of some sheaf space of po-quasigroups over X.

Especially, if $\mathcal{Q}$ is an l-quasigroup and θ_x are directed convex congruence relations on $\mathcal{Q}$ satisfying (i) and (ii), then $\mathcal{Q}$ can be o-embedded into an l-quasigroup of the continuous global sections of some sheaf space of l-quasigroups over X.

Construction of a sheaf space from theorem

$E = \bigcup_{x \in X} E_x$, where $E_x = \mathcal{Q}/\theta_x \times \{x\}$.

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Consider the finest topology τ on E such that each $\widehat{g}$ is continuous.

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Then

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Each fibre E_x is a po-quasigroup under the operations

$\widehat{g}(x) \cdot \widehat{h}(x) = (\widehat{gh})(x);\ \ (\widehat{g}(x)/\widehat{h}(x) = (\widehat{g/h})(x);\ \ \widehat{g}(x)\backslash\widehat{h}(x) = (\widehat{g\backslash h})(x)$

and the partial order

$\widehat{g}(x) \leq \widehat{h}(x)$ iff there exist $g' \in [g]\theta_x, h' \in [h]\theta_x$ such that $g' \leq h'$.

(E, X, σ) is a sheaf space of po-quasigroups.

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Let $\mathcal{R}$ be a po-quasigroup of all continuous global sections of (E, X, σ).

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The map

$$\Phi : \mathcal{Q} \rightarrow \mathcal{R};\ g \mapsto \widehat{g}.$$

is an o-embedding of $\mathcal{Q}$ into $\mathcal{R}$ (Φ preserves the quasigroup operations and $g \leq h \Leftrightarrow \widehat{g} \leq \widehat{h}$.).

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Let (E, X, σ) be the sheaf space of po-quasigroups constructed above. Let X be a Hausdorff space. Then E is a Hausdorff space if for all $g, h \in Q$, the set $U_{gh} = \{x \in X \mid [g]\theta_x = [h]\theta_x\}$ is open and also close in X.

A representable l-group can be represented as sections of a Hausdorff sheaf space of l-groups.

A representable l-group with a strong order unit can be l-embedded into a Hausdorff sheaf space of l-groups on a compact space.

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