Matrices in modular lattices

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G. Czédli and B. S.: *The ring of an outer von Neumann frame in modular lattices*, Algebra Universalis, accepted

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Let L be a *bounded modular* lattice. For $\vec{a} = (a_1, \ldots, a_m) \in L^m$ and $\vec{c} = (\ldots, c_{ij}, \ldots) \in L^{m(m-1)}$ $(i \neq j)$ we say that $(\vec{a}, \vec{c})$ is a *(spanning von Neumann) m-frame* of L, if:

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1. $\langle a_1, \ldots, a_m \rangle \leq L$ is a Boolean sublattice ($\cong \mathbf{2}^m$) with atoms a_i;
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3. $\langle a_j,\, c_{jk},\, a_k \rangle$ is an M_3 for $j \neq k$ and
4. $c_{ik} = c_{ki} = (c_{ij} + c_{jk})(a_i + a_k)$ for distinct i, j, k.

Example (canonical m-frame)

Let $1 \in R$ be a ring. Let e_i denotes the vector $(0, \ldots, 1, \ldots, 0) \in R^m$. Then $a_i = Re_i$ and $c_{ij} = R(e_i - e_j)$ form an m-frame of the submodule lattice $\mathrm{Sub}(R^m)$.

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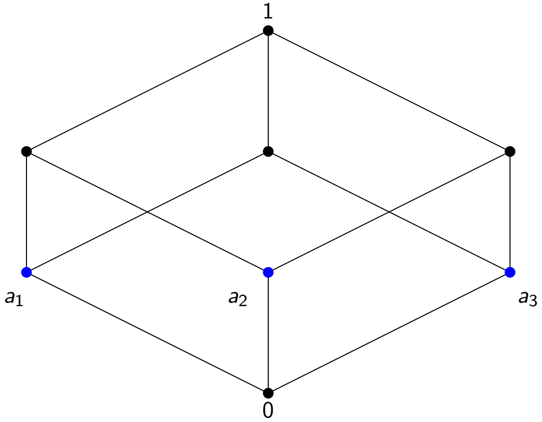

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- where we use the following *projectivities*:

$$R\big(\begin{smallmatrix} i & j \\ k & j \end{smallmatrix}\big) : [0, a_i + a_j] \to [0, a_k + a_j], \quad x \mapsto (x + c_{ik})(a_k + a_j),$$
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This $R\langle i,j\rangle$ is called the *coordinate ring* of the frame.

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- $R\langle i,j\rangle \cong R\langle i',j'\rangle$ for every $i' \neq j'$.

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For more details about frames and its coordinate rings, see the papers of

- J. von Neumann [8],
- B. Artmann [1],
- R. Freese [5] and [6],
- A. Day and D. Pickering [4] and
- C. Herrmann [7].

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Then

- *If $n \geq 2$, then there is a ring S^* such that $R^* \cong M_n(S^*)$.*
- *If $n \geq 4$ or L is Arguesian and $n \geq 3$, then we can choose S^* as the coordinate ring of $(\vec{u}, \vec{v})$.*

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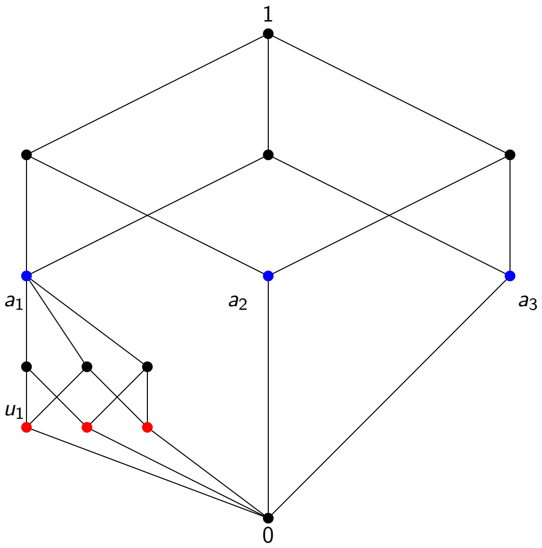
- Its coordinate ring S^* is isomorphic to S, hence $R^* \cong M_n(S^*)$.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a_1
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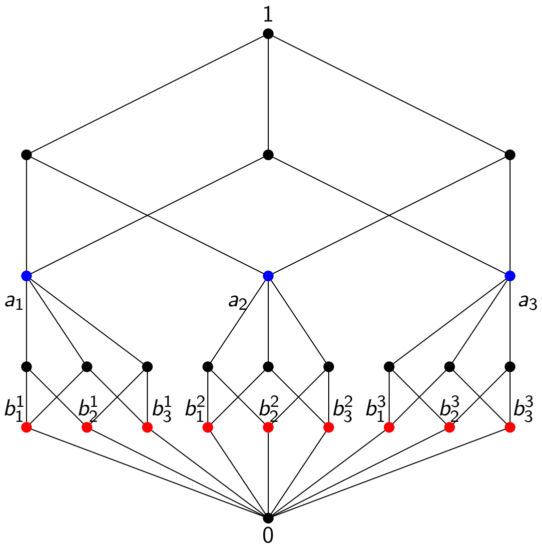
1
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b_1^1 b_2^1 b_3^1 b_1^2 b_2^2 b_3^2 b_1^3 b_2^3 b_3^3
0

The details of the next definition can be find in Czédli [2].

Definition

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- product frame $(\vec{b}, \vec{d})$

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The projectivities and coordinate rings in the product frame:

$$S\left\langle\begin{smallmatrix} p & q \\ i & j \end{smallmatrix}\right\rangle = \{x \in L : xb_j^q = 0,\ x + b_j^q = b_i^p + b_j^q\},$$

$$S\left(\begin{smallmatrix} pi & qj \\ pi & rk \end{smallmatrix}\right) : S\left\langle\begin{smallmatrix} p & q \\ i & j \end{smallmatrix}\right\rangle \to S\left\langle\begin{smallmatrix} p & r \\ i & k \end{smallmatrix}\right\rangle, \quad x \mapsto (x + d\,{}^{q\,r}_{j\,k})(b_i^p + b_k^r),$$

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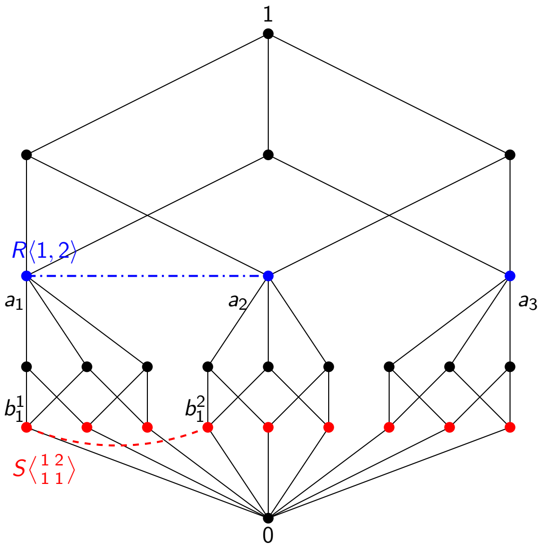
1
R⟨1, 2⟩
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b12
S⟨1 2 / 1 1⟩
0

$$M_3 = \begin{bmatrix} & S\left\langle\begin{smallmatrix}1 & 2\\ 1 & 1\end{smallmatrix}\right\rangle & \times & S\left\langle\begin{smallmatrix}1 & 2\\ 1 & 2\end{smallmatrix}\right\rangle & \times & S\left\langle\begin{smallmatrix}1 & 2\\ 1 & 3\end{smallmatrix}\right\rangle & \times \\ \times & S\left\langle\begin{smallmatrix}1 & 2\\ 2 & 1\end{smallmatrix}\right\rangle & \times & S\left\langle\begin{smallmatrix}1 & 2\\ 2 & 2\end{smallmatrix}\right\rangle & \times & S\left\langle\begin{smallmatrix}1 & 2\\ 2 & 3\end{smallmatrix}\right\rangle & \times \\ \times & S\left\langle\begin{smallmatrix}1 & 2\\ 3 & 1\end{smallmatrix}\right\rangle & \times & S\left\langle\begin{smallmatrix}1 & 2\\ 3 & 2\end{smallmatrix}\right\rangle & \times & S\left\langle\begin{smallmatrix}1 & 2\\ 3 & 3\end{smallmatrix}\right\rangle & \end{bmatrix}$$

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$$\varphi_{ij}\colon R\langle 1,2\rangle \to S\left\langle\begin{smallmatrix}1&2\\i&j\end{smallmatrix}\right\rangle, \quad x \mapsto x_{ij} = (x + \sum_{i\neq j} b_i^2)(b_i^1 + b_j^2)$$

$$\psi\colon M_n \to R\langle 1,2\rangle, \quad (e_{ij} : i,j \leq n) \mapsto \prod_k (\sum_i e_{ik} + B_k^2).$$

Thank you!

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