Extremal states on bounded residuated ℓ-monoids with general comparability

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Bounded $R\ell$-monoids

Definition

A bounded $R\ell$-monoid is an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$ satisfying:

- $(M; \odot, 1)$ is a monoid;
- $(M; \vee, \wedge, 0, 1)$ is a bounded lattice;
- $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$;
- $(x \rightarrow y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$.

- $(M; \vee, \wedge)$ is distributive,
- $\odot$ distributes over the lattice operations,
- bounded $R\ell$-monoids – a variety of algebras;

In what follows, an $R\ell$-monoid is a bounded $R\ell$-monoid.

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A bounded $R\ell$-monoid is an algebra $M = (M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$ satisfying:

- $(M; \odot, 1)$ is a monoid;
- $(M; \vee, \wedge, 0, 1)$ is a bounded lattice;
- $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$;
- $(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$.

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A remark to terminology

- bounded residuated lattices satisfying the condition of divisibility,
- bounded integral generalized BL-algebras,
- pseudo BL-algebras as the subvariety of FL-algebras satisfying weakening.

Additional unary operations:

$$x^{-} := x \to 0$$

$$x^{\sim} := x \rightsquigarrow 0$$

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Special cases of $R\ell$-monoids

An $R\ell$-monoid M is

- a pseudo *BL*-algebra iff
$$(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x);$$
- a *GMV*-algebra (pseudo *MV*-algebra) iff
$$x^{-\sim} = x = x^{\sim -};$$
- a Heyting algebra iff
$$x \odot x = x \quad (\odot = \wedge).$$

Particular classes of $R\ell$-monoids

If $\odot$ is commutative then an $R\ell$-monoid is called commutative.
In such a case, $\rightarrow = \rightsquigarrow$ and $^- = {}^\sim$.

Definition

An $R\ell$-monoid is called *good* if it satisfies

$$x^{-\sim} = x^{\sim -}.$$

We define the binary operation $\oplus$:

$$x \oplus y := (y^- \odot x^-)^\sim.$$

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Filters

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A non-empty subset F of an $R\ell$-monoid M is called a *filter* of M if

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(F2) $x \in F,\ y \in M,\ x \leq y$ imply $y \in F$.

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The set $\mathcal{F}(M)$ of all filters of M is a complete lattice.

If $\emptyset \neq X \subseteq M$ and $F(X)$ is the filter of M generated by X, then

$$F(X) = \{x \in M : \ x \geq a_1 \odot \cdots \odot a_n, \ \text{where} \ a_1, \ldots, a_n \in X, \ n \in \mathbb{N}\}.$$

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Normal filters and congruences

Definition

A filter F is called *normal* if for each $x, y \in M$

(F3) $\quad x \to y \in F \iff x \rightsquigarrow y \in F.$

normal filters of $M \quad \longleftrightarrow \quad$ kernels of congruences on M

$$\begin{aligned} \langle x, y \rangle \in \Theta(F) &\iff (x \to y) \wedge (y \to x) \in F \\ &\iff (x \rightsquigarrow y) \wedge (y \rightsquigarrow x) \in F. \end{aligned}$$

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Boolean elements

Let $M = (M;\ \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ be a bounded $R\ell$-monoid.

Definition

An element $a \in M$ is called *Boolean* if it has a complement in M, i.e. there is an element $x' \in M$ such that $x \wedge x' = 0$ and $x \vee x' = 1$.

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Denote by $B(M)$ the set of all Boolean elements of M.

$(B(M);\ \vee, \wedge, ', 0, 1)$

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- For any $a \in B(M)$ and $x \in M$: $a \odot x = a \wedge x = x \odot a$.

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Supportive propositions

For any $a \in B(M)$:

$$M_a := \{x \in M : \ 0 \leq x \leq a\}.$$

$\odot_a$, $\wedge_a$, $\vee_a$ ...... the restrictions of $\odot, \wedge, \vee$ from M on M_a;

For every $x, y \in M_a$: $x \rightarrow_a y := (x \rightarrow y) \wedge a$, $x \rightsquigarrow_a y := (x \rightsquigarrow y) \wedge a$.

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(a) $M_a = (M_a; \odot_a, \vee_a, \wedge_a, \to_a, \rightsquigarrow_a, 0, a)$ *is a bounded $R\ell$-monoid.*

(b) *The mapping $p_a : M \longrightarrow M_a$ such that $p_a(x) := x \wedge a$, for each $x \in M$, is a surjective homomorphism of M onto M_a.*

(c) *M is isomorphic with $M_a \times M_{a'}$.*

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General comparability property

Dvurečenskij, Rachůnek: *Bounded commutative residuated ℓ-monoids with general comparability and states.* Soft Comput. 10 (2006).

Definition

A bounded $R\ell$-monoid M satisfies *general comparability* if for every $x, y \in M$ there is $a \in B(M)$ such that

$$p_a(x) \leq p_a(y) \text{ and } p_{a'}(x) \geq p_{a'}(y).$$

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The idea

- Two elements of M need not be in general comparable in M.
- The coordinates of elements $x = (x \wedge a, x \wedge a')$, $y = (y \wedge a, y \wedge a')$ can be compared in $[0, a]$ and $[0, a']$, respectively.

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Examples

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- Every linearly ordered bounded $R\ell$-monoid satisfies general comparability.
- The direct product of an arbitrary system of linearly ordered bounded $R\ell$-monoids satisfies general comparability.

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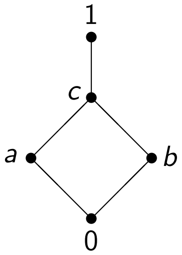
Let $M = \{0, a, b, c, 1\}$ be the lattice with the given diagram, $\odot = \wedge$, $\to = \rightsquigarrow$, and $\to$ be defined by the table.

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1



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b	a	a	1	1	1
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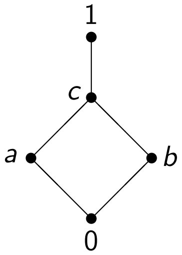

Then $R\ell$-monoid $M = (M;\ \vee, \wedge, \odot, \rightarrow, 0, 1)$ does not satisfy general comparability.

A glimpse of states on fuzzy structures

Mundici, D.: *Averaging the truth-value in Łukasiewicz logic.* Studia Logica 55 (1995).

- States establish measures on their associated *MV*-algebras which generalized the usual probability measures on Boolean algebras.

States on *MV*-algebras

A *state* on an *MV*-algebra M is a mapping $s: M \longrightarrow [0,1]$ $(\subset \mathbb{R})$ such that

(1) $s(x \oplus y) = s(x) + s(y)$ if $x \odot y = 0$ $(\iff y \leq x^{-})$,

(2) $s(1) = 1$.

A glimpse of states on fuzzy structures

Riečan, B.: *On the probability on BL-algebra.* Acta Math. Nitra 4 (2000).

Riečan states

A *Riečan state* on a BL-algebra M is a mapping $s: M \longrightarrow [0,1]$ such that

(R1) $s(x \oplus y) = s(x) + s(y)$ if $y^{--} \leq x^{-}$,

(R2) $s(1) = 1$.

A glimpse of states on fuzzy structures

Dvurečenskij, A.: *States on pseudo MV-algebras*, Stud. Logica 68 (2001).

States on GMV-algebras

A *state* on an GMV-algebra (pseudo MV-algebra) M is a mapping $s: M \longrightarrow [0,1]$ such that

(1) $s(x \oplus y) = s(x) + s(y)$ if $x \odot y = 0$ ($\Longleftrightarrow$ $y \leq x^{\sim}$ $\Longleftrightarrow$ $x \leq y^{-}$),

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Georgescu, G.: *Bosbach states on fuzzy structures*, Soft Comput. 8 (2004).

- He extended the notion of a Riečan state for good pseudo *BL*-algebras.
- He introduced a Bosbach state not using orthogonal elements, so it is applicable also for non-commutative fuzzy structures, which are not good.

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Dvurečenskij, A., Rachůnek, J.: *On Riečan and Bosbach states for bounded non-commutative $R\ell$-monoids,* Math. Slovaca 56 (2006).

- For good bounded $R\ell$-monoids Riečan and Bosbach states coincide.

Bosbach states

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A *Bosbach state* (further simply a *state*) on M is a mapping $s: M \longrightarrow [0,1]$ such that for any $x, y \in M$,

(B1) $s(x) + s(x \rightarrow y) = s(y) + s(y \rightarrow x)$;

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Kernel of s

If s is a state on a bounded $R\ell$-monoid M, set

$$\mathrm{Ker}(s) := \{x \in M : s(x) = 1\}.$$

- $\mathrm{Ker}(s)$ is a proper normal filter of M.
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Extremal states

The set $\mathcal{S}(M)$ of all states on M is a convex set, i.e.,
if $s_1,\ s_2 \in \mathcal{S}(M)$ and $\lambda \in [0,1]$, then $s = \lambda s_1 + (1-\lambda)s_2 \in \mathcal{S}(M)$.

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A state $s \in \mathcal{S}(M)$ is called *extremal* if the equality $s = \lambda s_1 + (1-\lambda)s_2$, where $s_1,\ s_2 \in \mathcal{S}(M)$ and $\lambda \in (0,1)$, implies $s = s_1 = s_2$.

Proposition

Let s be a state on a bounded $R\ell$-monoid M and let $K = B(M) \cap \mathrm{Ker}(s)$. Then it holds:

(a) *If s is extremal, then K is a maximal filter of $B(M)$.*

(b) *If s has the property that $t \in \mathcal{S}(M)$ and $\mathrm{Ker}(t) \supseteq K$ imply $t = s$, then s is extremal.*

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General comparability and states

Theorem

Let a bounded $R\ell$-monoid M satisfy general comparability and let K be a maximal filter of $B(M)$ such that the filter $F(K)$ of the $R\ell$-monoid M generated by K is normal.
Then there exists a unique state s on M such that $B(M) \cap \mathrm{Ker}(s) = K$.
Moreover, the state s is extremal.

General comparability and states

Corollary

If M is a bounded $R\ell$-monoid satisfying general comparability and if there is at least one maximal filter K of $B(M)$ such that the filter $F(K)$ of the $R\ell$-monoid M generated by K is normal, then $\mathcal{S}(M) \neq \emptyset$.

Corollary

Let M be a bounded $R\ell$-monoid satisfying general comparability such that for every maximal filter K of $B(M)$, the filter $F(K)$ of M generated by K is normal. If the set of extremal states on M is finite, then every state on $B(M)$ can be extended to a state on M.

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General comparability and maximal filters of $B(M)$

Proposition

Let M be a bounded $R\ell$-monoid. Let $\bigcap_K F(K)$ be the intersection of all filters $F(K)$ of M generated by maximal filters K of $B(M)$. Then $\bigcap_K F(K) = \{1\}$.

Theorem

(a) *If a bounded $R\ell$-monoid M satisfies general comparability, K is a maximal filter of $B(M)$ and the filter $F(K)$ of M generated by K is normal, then the quotient $R\ell$-monoid $M/F(K)$ is linearly ordered.*

(b) *If, moreover, the filter $F(K)$ is normal in M for every maximal filter K of $B(M)$, then M is a pseudo-BL-algebra which is a subdirect product of the pseudo-BL-algebras $M/F(K)$.*

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