ON RESIDUATION SUBREDUCTS OF POCRIGS

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2 Some implicative algebras
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1. LEFT RESIDUATED GROUPOIDS

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This condition is equivalent to the following four:

$x \leq y \to xy$,

$(x \to y)x \leq y$,

if $x \leq y$, then $xz \leq yz$,

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If $\cdot$ satisfies also the condition

if $x \leq y$, then $zx \leq zy$,

then the system $(P, \cdot, \rightarrow)$ is called a *partially ordered left residuated groupoid*.

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A polrig is idempotent iff its multiplication is the meet operation.
So, ipolrigs are just implicative (or Brouwerian, or relatively pseudocomlemented) semilattices.

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- " -	pocrims	"	BCK-algebras (M.Palasinski, 1982; I.Fleisher, 1988; H.Ono, Y.Komori, 1985)
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$\boxed{\text{What are the residuation subreducts of polrigs, pocrigs and ipolrigs?}}$

!! ipolrig = ipolrim

2. SOME IMPLICATIVE ALGEBRAS

An *implicative algebra* is an algebra $(A, \rightarrow, 1)$, where

- A is a poset with the greatest element 1,
- $\rightarrow$ is a binary operation such that

  $x \leq y$ if and only if $x \rightarrow y = 1$.

(H.Rasiowa, 1974)

An implicative algebra is said to be *normal* (*NI-algebra*) if the identity $1 \rightarrow x = x$ holds in it.

quasi-BCC-algebra: NI if $x \leq y$, then $z \to x \leq z \to y$ if $x \leq y$, then $y \to z \leq x \to z$
quasi-BCK-algebra: qBCC $x \leq (x \to y) \to y$
positive implicative quasi-BCK-algebra or *quasi-Hilbert algebra*: qBCK if $x \leq x \to y$, then $x \leq y$

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BCK $=$ qBCK $+$ $[x \to (y \to z) = y \to (x \to z)]$,
H $=$ qH $+$ $[x \to (y \to z) = y \to (x \to z)]$.

BCC $=$ qBCC $+$?

Example. Let $(A, \cdot, \to, 1)$ be a polrig. Then the reduct $(A, \to, 1)$ is

a qBCC algebra	– always
a qBCK algebra	iff the polrig is commutative,
a qH algebra	iff the polrig is idempotent,
a BCC algebra	iff the polrig is left semi-associative:
	$xy \cdot z \leq x \cdot yz.$

It follows that a polrig is an implicative semilattice if and only if its $(\to, 1)$-reduct is a qH-algebra.

3. MAIN RESULT

Theorem.

r-subreducts of	polrigs	are	the qBCC-algebras,
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Unanswered question:

Are r-subreducts of ipolrigs the qH-algebras?

(If yes, then $qH = H$.)

4. BETWEENNESS FRAMES [after M. Dunn, 1994]

A *frame* is a triple (K, R, β), where

K is a set,

R is a binary relation on K,

β is a ternary relation on K

xRy reads as "y is accessible from x",

βxyz reads as "z is between x and y",

and

R is reflexive,

$\forall xyz\, x'y'z'\, (\beta xyz \,\&\, xR\,x' \,\&\, yR\,y' \,\&\, z'R\,z \supset \beta x'y'z')$.

The relation β in a frame is said to be

reflexive if

$\forall x\, \beta xxx$,

balanced if

$\forall xy\, (\beta xxy \supset yR\, x)$,

symmetric if

$\forall xyz\, (\beta xyz \supset \beta yxz)$,

a *left Pasch relation* if:

$\forall xyz \forall u\, [\, \exists v\, (\beta xyv \,\&\, \beta vzu) \supset \exists w(\beta yzw \,\&\, \beta xwu)\,]$,

a *right Pasch relation* if:

$\forall xyz \forall u\, [\, \exists v\, (\beta yxv \,\&\, \beta zvu) \supset \exists w(\beta zyw \,\&\, \beta wxu)\,]$.

A frame is *assertional*, if

$$\forall uv\,(uR\,v \equiv \exists x\,\beta xvu),$$

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An element a of a frame is said to be *initial* if the following holds:

$$\forall y\ \ aR\,y,$$
$$\forall uv\,(uR\,v \equiv \beta avu),$$
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An *initialized frame* is a frame together with a selected initial element.

Example 1: Betweenness spaces I.

K − the set of points of a plane,
R − equality $=$,
β − the ordinary betweenness relation for points.

The system $(K, =, \beta)$ is a frame with a reflexive, balanced and symmetric Pasch relation β.

Pasch axiom in plane:

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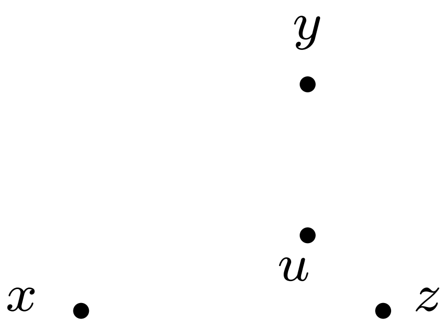

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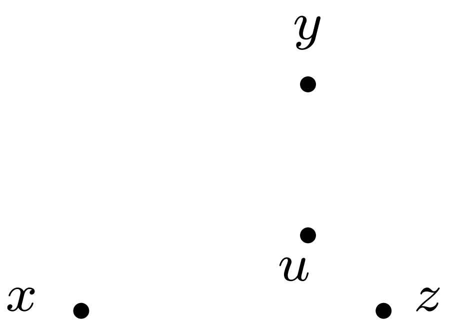

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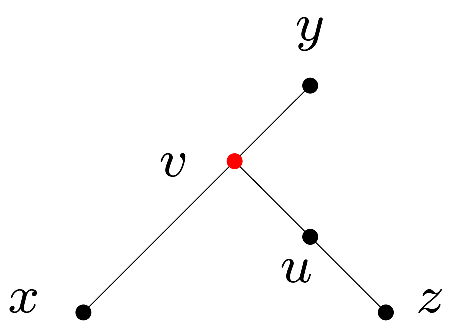

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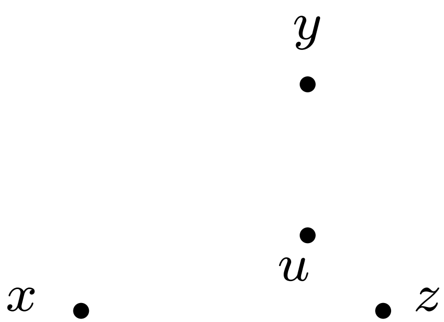

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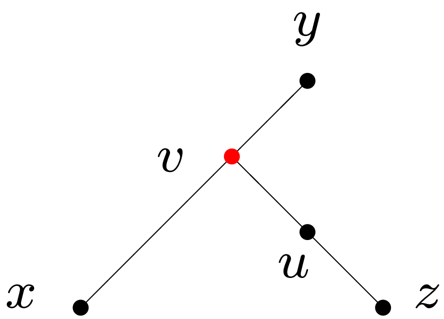

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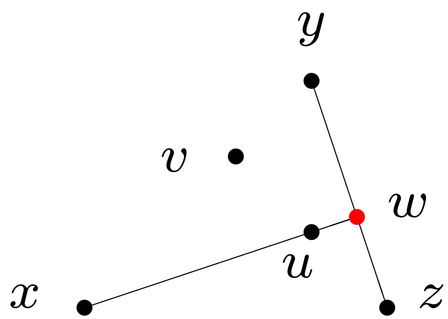

Example 2: Betweenness spaces II.

K – the set of closed regions on a plane,
R – inclusion $\subseteq$,
B – betweenness relation for regions,
ϕ – the empty region.

A region z is said to be *between* x and y if z is included in the convex hull of $x \cup y$.

The system $(K, \subseteq, B, \phi)$ is an initialized frame with a reflexive and symmetric Pasch relation B.

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Then the following assertions are equivalent:

(a) $(K, \rightarrow, 1)$ is a qBCC algebra and, for all x, y, z,

$\beta xyz :\equiv x \leq y \rightarrow z$,

(b) $(K, \geq, \beta, 1)$ is an initialized frame and, for all y, z,

$y \rightarrow z = \max(x\colon \beta xyz)$.

The frame in (b) is called the *characteristic frame* of the qBCC-algebra.

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If (a) and (b) hold, then

K is a	qBCK-algebra	iff	β is	symmetric,
- " -	qH-algebra	iff	-"-	balanced.

Example 3: continuation

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If (a) and (b) hold, then
K is a polrim iff β is a left and right Pasch relation,
K is a pocrim iff β is a symmetric Pasch relation.

Example 4: Canonic frames

Suppose that (K, R, β, a) is an initialized frame.

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$H(K) := \{X\colon X$ is hereditary$\}$.

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The system $(H(K), \supseteq, B)$, where $B \subseteq H(K)^3$ is defined by

$$\begin{aligned} B(X, Y, Z) &:\equiv (\forall x \in X)\,(\forall y \in Y)\,\forall z\,(\beta xyz \supset z \in Z) \\ &\equiv \forall z\,[\,(\exists x \in X)\,(\exists y \in Y)\,\beta xyz \supset z \in Z\,], \end{aligned}$$

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If K is the characteristic frame of a qBCC-algebra, then $H(K)$ is called the *canonic* frame for the latter.

5. PROOF

Proposition 2. Let (K, R, β) be a frame. For $X, Y, Z \subseteq K$, put

$$\begin{aligned} X \circ Y &:= \{z\colon (\exists x \in X)(\exists y \in Y)\, \beta xyz\}, \\ Y \Rightarrow Z &:= \{x\colon \forall z (\forall y \in Y)\, (\beta xyz \supset z \in Z)\}, \\ \mathbb{1} &:= K. \end{aligned}$$

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Then

(a) $(\mathcal{P}(K), \circ, \Rightarrow, \mathbb{1})$ is a partially ordered residuated groupoid with $\mathbb{1}$ the largest element,

(b) the operation $\circ$ is

idempotent	if β is reflexive,
commutative	if β is symmetric,
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Proposition 3. $H(K)$ contains $\mathbb{1}$ and is closed under $\circ$ and $\Rightarrow$. Hence, $(H(K), \circ, \Rightarrow, \mathbb{1})$ also is a residuated groupoid with the greatest element. It is integral if and only if K is assertional.

Now, let $(A, \to, 1)$ be a qBCC-algebra.

Move to its characteristic (initialized) frame $(A, \geq, \beta, 1)$.

Move further to its canonic (assertional) frame $(K, \supseteq, B)$ with $K = H(A)$.

Construct the associated polrig $(H(K), \circ, \Rightarrow, \mathbb{1})$.

(If A is a qBCK-algebra, then $H(K)$ is a pocrig.)

Proposition 4. The mapping h: $A \to H(K)$ defined by

$h(a) := \{F \in H(K) \colon a \in F\}$

is an embedding of A into the reduct $(H(K), \Rightarrow, \mathbb{1})$.