Weak Homomorphisms for (F_1, F_2)-systems

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In 1960, E. Marczewski proposed the concept of weak homomorphisms for non-indexed algebras and in 1980, K.Glazek proposed this concept for indexed algebras.

Let $\underline{A} = (A; (f_i^{\underline{A}})_{i \in I})$ and $\underline{B} = (B; (g_j^{\underline{B}})_{j \in J})$ be algebras of types τ_1 and τ_2. A mapping $\varphi : A \to B$ is said to be a weak homomorphism from $\underline{A}$ to $\underline{B}$ if for each n_i-ary fundamental operation $f_i^{\underline{A}}$ there exists an n_i-ary term operation $s^{\underline{B}}$ of algebra $\underline{B}$ such that

$$\varphi(f_i^{\underline{A}}(a_1, \ldots, a_{n_i})) = s^{\underline{B}}(\varphi(a_1), \ldots, \varphi(a_{n_i}))$$

for all $a_1, \ldots, a_{n_i} \in A$ and for each n_j-ary fundamental operation $g_j^{\underline{B}}$ there exists an n_j-ary term operation $t^{\underline{A}}$ of algebra $\underline{A}$ such that

$$\varphi(t^{\underline{A}}(a_1, \ldots, a_{n_j})) = g_j^{\underline{B}}(\varphi(a_1), \ldots, \varphi(a_{n_j}))$$

for all $a_1, \ldots, a_{n_j} \in A$.

Let $F : \mathcal{S}et \to \mathcal{S}et$ be a functor. An F-algebra $\mathcal{A}$ is a pair of a set A and a mapping $\alpha_A : F(A) \to A$.

Every algebra of type τ can be regarded as an F-algebra where this functor F is called an algebra functor with respect to a type τ be defined as follow:

An algebra functor with respect to a type $\tau = (n_i)_{i \in I}$ is a functor $F^\tau : \mathcal{S}et \to \mathcal{S}et$ which is defined by:

for each set X , $F^\tau(X) = \sum\limits_{i \in I} X^{n_i} = \bigcup\limits_{i \in I} \{(i, \overline{x}) \mid \overline{x} \in X^{n_i}\}$

for each mapping $\varphi : X \to Y$, $F^\tau(\varphi) : \sum\limits_{i \in I} X^{n_i} \to \sum\limits_{i \in I} Y^{n_i}$

$$\text{by } (i, (x_1, \ldots, x_{n_i})) \longmapsto (i, (\varphi(x_1), \ldots, \varphi(x_{n_i}))).$$

By the definition of algebra functor, we know that

For each algebra $\underline{A} = (A, (f_i^{\underline{A}})_{i \in I})$ of type τ, there is an F^τ-algebra $\mathcal{A} = (A; \alpha_A)$ where $\alpha_A : F^\tau(A) \to A$ by $\alpha_A(i, (a_1, \ldots, a_{n_i})) = f_i^{\underline{A}}(a_1, \ldots, a_{n_i})$ for all $(i, (a_1, \ldots, a_{n_i})) \in F^\tau(A)$.

For each F^τ-algebra $\mathcal{A} = (A; \alpha_A)$, there is an algebra $\underline{A} = (A, (f_i^{\underline{A}})_{i \in I})$ of type τ where for each $i \in I$, $f_i^{\underline{A}} : A^{n_i} \to A$ by $f_i^{\underline{A}}(a_1, \ldots, a_{n_i}) = \alpha_A(i, (a_1, \ldots, a_{n_i}))$ for all $a_1, \ldots, a_{n_i} \in A$.

Moreover, the category $Alg\tau$ with objects as algebras and morphisms as homomorphism and the category Set^{F^τ} with objects as F^τ-algebra and morphisms as homomorphisms are isomorphic.

F.M.Schneider generalized concept of K.Glazek to weak homomorphisms for F-algebras.

F-algebra can be regarded as an (F_1, F_2)-system where $F_1 = F$ and F_2 is an identity functor. We will generalize Schneider's idea to weak homomorphisms for (F_1, F_2)-systems.

Outline

1. (F_1, F_2)-Systems

2. Weak Homomorphisms for (F_1, F_2)-Systems

1. (F_1, F_2)-Systems

Definition 1.1

Let $F_1, F_2 : \mathcal{S}et \to \mathcal{S}et$ be functors. An (F_1, F_2)-**system** $\mathcal{A}$ is a pair of a set A and a mapping $\alpha_A : F_1(A) \to F_2(A)$.

In case F_1 is the identity functor, (F_1, F_2)-system $\mathcal{A}$ is said to be an F_2-coalgebra.

In case F_2 is the identity functor, (F_1, F_2)-system $\mathcal{A}$ is said to be an F_1-algebra.

Definition 1.2

Let $\mathcal{A} = (A; \alpha_A)$ and $\mathcal{B} = (B; \alpha_B)$ be (F_1, F_2)-systems. A mapping $\varphi : A \to B$ is called a **homomorphism** from $\mathcal{A}$ to $\mathcal{B}$, written as $\varphi : \mathcal{A} \to \mathcal{B}$, if

$$F_2(\varphi) \circ \alpha_A = \alpha_B \circ F_1(\varphi).$$

$$\begin{tikzcd}
F_1(A) \arrow[r, "F_1(\varphi)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(=)"] & F_1(B) \arrow[d, "\alpha_B"] \\
F_2(A) \arrow[r, "F_2(\varphi)"'] & F_2(B)
\end{tikzcd}$$

Theorem 1.3[2], [3]

Let $\mathcal{A} = (A; \alpha_A), \mathcal{B} = (B; \alpha_B)$ and $\mathcal{C} = (C; \alpha_C)$ be (F_1, F_2)-systems and let $\psi : \mathcal{A} \to \mathcal{B}$ and $\varphi : \mathcal{B} \to \mathcal{C}$ be homomorphisms. Then

1) The identity mapping $id_A : A \to A$ is a homomorphism from $\mathcal{A}$ to $\mathcal{A}$.
2) The composition function $\varphi \circ \psi : A \to C$ is a homomorphism from $\mathcal{A}$ to $\mathcal{C}$.

The class of all (F_1, F_2)-systems together with homomorphisms forms a category, written as $\mathcal{S}et_{(F_1,F_2)}$.

Theorem 1.4[3](The Factorization Theorem)

Let $\mathcal{A} = (A; \alpha_A)$ and $\mathcal{B} = (B; \alpha_B)$ be (F_1, F_2)-systems and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism. If $\varphi = \psi \circ \pi$ is a factorization where $\pi : A \twoheadrightarrow Q$ and $\psi : Q \rightarrowtail B$, then there is a unique mapping $\alpha_Q : F_1(Q) \to F_2(Q)$ such that π and ψ are homomorphisms.

$$\begin{tikzcd}
F_1(A) \arrow[r, "F_1(\pi)"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "(=)"] & F_1(Q) \arrow[r, "F_1(\psi)"] \arrow[d, dashed, "\alpha_Q"] \arrow[dr, phantom, "(=)"] & F_1(B) \arrow[d, "\alpha_B"] \\
F_2(A) \arrow[r, "F_2(\pi)"'] & F_2(Q) \arrow[r, "F_2(\psi)"'] & F_2(B)
\end{tikzcd}$$

Definition 1.5

Let $\mathcal{A} = (A; \alpha_A)$ be an (F_1, F_2)-system. A subset S of A is said to be **open** in $\mathcal{A}$ if there is a mapping $\alpha_S : F_1(S) \to F_2(S)$ such that the embedding $\subseteq_S^A : S \hookrightarrow A$ is a homomorphism, and $\mathcal{S} = (S; \alpha_S)$ is called an (F_1, F_2)-**subsystem** of $\mathcal{A}$, written as $\mathcal{S} \preceq \mathcal{A}$.

$$\begin{tikzcd}
F_1(S) \arrow[r, "F_1(\subseteq_S^A)"] \arrow[d, "\alpha_S"'] \arrow[dr, phantom, "(=)"] & F_1(A) \arrow[d, "\alpha_A"] \\
F_2(S) \arrow[r, "F_2(\subseteq_S^A)"'] & F_2(A)
\end{tikzcd}$$

Proposition 1.6[3]

Let $\mathcal{A} = (A; \alpha_A)$ and $\mathcal{B} = (B; \alpha_B)$ be (F_1, F_2)-systems and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism.

(1) If $S \subseteq A$ is open in $\mathcal{A}$, then $\varphi[S]$ is open in $\mathcal{B}$.

(2) If $R \subseteq B$ is open in $\mathcal{B}$ and F_2 preserves pullbacks, then $\varphi^{-1}[R]$ is open in $\mathcal{A}$.

On the category $\mathcal{S}et$, we know that for each mapping $\varphi : A \to B$, $Ker\varphi$ together with canonical projections $\pi_1, \pi_2 : Ker\varphi \to A$ forms a pullback of φ and φ.

$$
\begin{tikzcd}
 & A \arrow[dl, "\varphi"'] & & \\
B & & Ker\varphi \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_2"] \arrow[u, phantom, "(=)"] \arrow[d, phantom, "(=)"] & Q \arrow[l, dashed, "\exists ! h"'] \arrow[ull, "q_1"'] \arrow[dll, "q_2"] \\
 & A \arrow[ul, "\varphi"] & &
\end{tikzcd}
$$

Definition 1.7

A functor $F : \mathcal{S}et \to \mathcal{S}et$ **weakly preserves kernels** if for each mapping $\varphi : A \to B$, $F(Ker\varphi)$ together with $F(\pi_1), F(\pi_2)$ where $\pi_1, \pi_2 : Ker\varphi \to A$ are canonical projections, forms a pullback of $F(\varphi)$ and $F(\varphi)$.

Proposition 1.8

Let $\mathcal{A} = (A; \alpha_A)$ and $\mathcal{B} = (B; \alpha_B)$ be (F_1, F_2)-systems.
If φ is a homomorphism from $\mathcal{A}$ to $\mathcal{B}$
and F_2 weakly preserves kernels and preserves products,
then $Ker\varphi$ is open in $\mathcal{A} \times \mathcal{A}$.

Proposition 1.9

Let $\mathcal{A} = (A; \alpha_A)$ be an (F_1, F_2)-system and let $\varphi : A \to B$ be a mapping.
If (1) φ is surjective and
(2) $\mathcal{A} \times \mathcal{A}$ exists and $Ker\varphi$ is open in $\mathcal{A} \times \mathcal{A}$ and
(3) F_1 weakly preserves kernels,
then there is a unique mapping $\alpha_B : F_1(B) \to F_2(B)$ such that φ is a homomorphism from $\mathcal{A}$ to $\mathcal{B} = (B; \alpha_B)$.

Proposition 1.10

Let $\mathcal{B} = (B; \alpha_B)$ be an (F_1, F_2)-system and let $\varphi : A \to B$ be a mapping.
If (1) φ is injective and
(2) $\varphi[A]$ is open in $\mathcal{B}$,
then there is a unique mapping $\alpha_A : F_1(A) \to F_2(A)$ such that φ is a homomorphism from $\mathcal{A} = (A; \alpha_A)$ to $\mathcal{B}$.

Let $\mathcal{A} = (A; \alpha_A)$ be an (F_1, F_2)-system and let I be an arbitrary set. If F_2 preserves products, then the direct power $\mathcal{A}^I$ exists and its universe is the direct power A^I of the universe of $\mathcal{A}$ in the category $\mathcal{S}et$.

Let $Sub(\mathcal{A})$ denote the set of all open sets in $\mathcal{A}$.

Definition 1.11

Let $\mathcal{A}_{12} = (A; \alpha_{12})$ and $\mathcal{A}_{34} = (A; \alpha_{34})$ be an (F_1, F_2)-system and an (F_3, F_4)-system, respectively. We say that the structural mappings α_{12} and α_{34} are **algebraically equivalent**, written as $\alpha_{12} \equiv \alpha_{34}$, if $Sub(\mathcal{A}_{12}^I) = Sub(\mathcal{A}_{34}^I)$ for all set I.

Proposition 1.12

Let $\mathcal{A}_{12} = (A; \alpha_{12})$ and $\mathcal{B}_{12} = (B; \beta_{12})$ be (F_1, F_2)-systems,
let $\mathcal{A}_{34} = (A; \alpha_{34})$ and $\mathcal{B}_{34} = (B; \beta_{34})$ be (F_3, F_4)-systems,
let φ be a homomorphism from $\mathcal{A}_{12}$ to $\mathcal{B}_{12}$
and a homomorphism from $\mathcal{A}_{34}$ to $\mathcal{B}_{34}$, and
let F_2 and F_4 preserve products and pullbacks.
(1) If $\beta_{12} \equiv \beta_{34}$ and φ is injective, then $\alpha_{12} \equiv \alpha_{34}$.
(2) If $\alpha_{12} \equiv \alpha_{34}$ and φ is surjective, then $\beta_{12} \equiv \beta_{34}$.

$$\begin{array}{ccc} \mathcal{A}_{12} = (A; \alpha_{12}) & & \mathcal{B}_{12} = (B; \beta_{12}) \\ & \xrightarrow{\quad\quad\varphi\quad\quad} & \\ \mathcal{A}_{34} = (A; \alpha_{34}) & & \mathcal{B}_{34} = (B; \beta_{34}) \end{array}$$

Definition 2.1

Let $\mathcal{A}_{12} = (A; \alpha_{12})$ be an (F_1, F_2)-system and let $\mathcal{B}_{34} = (B; \beta_{34})$ be an (F_3, F_4)-system. A mapping $\varphi : A \to B$ is called a **weak homomorphism** from $\mathcal{A}_{12}$ to $\mathcal{B}_{34}$ if for each factorization $\varphi = \psi \circ \pi$ where $\pi : A \twoheadrightarrow Q$ and $\psi : Q \rightarrowtail B$, there are mappings $\gamma_{12} : F_1(Q) \to F_2(Q)$ and $\gamma_{34} : F_3(Q) \to F_4(Q)$ such that

(i) $\gamma_{12} \equiv \gamma_{34}$,and

(ii) π is a homomorphism from $\mathcal{A}_{12}$ to $\mathcal{Q}_{12} = (Q; \gamma_{12})$,and

(iii) ψ is a homomorphism from $\mathcal{Q}_{34} = (Q; \gamma_{34})$ to $\mathcal{B}_{34}$.

By The Factorization Theorem, every homomorphism is a weak homomorphism. Then identity mapping is a weak homomorphism.

Proposition 2.2

Let $\mathcal{A}_{12} = (A; \alpha_{12})$ and $\mathcal{B}_{34} = (B; \beta_{34})$ be (F_1, F_2)-system and (F_3, F_4)-system, respectively and let $\varphi : A \to B$ be a mapping.
If F_2 and F_4 preserve products and pullbacks, and

there is a factorization $\varphi = \psi \circ \pi$ where $\pi : A \twoheadrightarrow Q$ and $\psi : Q \rightarrowtail B$ such that there are mappings $\gamma_{12} : F_1(Q) \to F_2(Q)$ and $\gamma_{34} : F_3(Q) \to F_4(Q)$ such that satisfy conditions (i), (ii) and (iii) in definition 2.1,

then φ is a weak homomorphism from $\mathcal{A}_{12}$ to $\mathcal{B}_{34}$.

Theorem 2.3

Let $\mathcal{A}_{12} = (A; \alpha_{12})$, $\mathcal{B}_{34} = (B; \alpha_{34})$ and $\mathcal{C}_{56} = (C; \alpha_{56})$ be (F_1, F_2)-system, (F_3, F_4)-system and (F_5, F_6)-system, respectively.
If (1) F_1 weakly preserves kernels and
(2) F_2, F_4 and F_6 preserve products and pullbacks and
(3) φ_1 is a weak homomorphisms from $\mathcal{A}_{12}$ to $\mathcal{B}_{34}$ and
(4) φ_2 is a weak homomorphisms from $\mathcal{B}_{34}$ to $\mathcal{C}_{56}$,
then the composition $\varphi_2 \circ \varphi_1 : A \to C$ is a weak homomorphism from $\mathcal{A}_{12}$ to $\mathcal{C}_{56}$.

Let $\mathcal{K}_1$ be the class of all $\mathcal{S}et$-endofunctors which weakly perserve kernels.

Let $\mathcal{K}_2$ be the class of all $\mathcal{S}et$-endofunctors which perserve products and pullbacks.

Then the class of all (F_1, F_2)-systems where $F_1 \in \mathcal{K}_1$ and $F_2 \in \mathcal{K}_2$, together with weak homomorphisms forms a category, written as $\mathcal{S}et_{(\mathcal{K}_1,\mathcal{K}_2)}$.

Proposition 2.4

Let $\mathcal{A}_{12} = (A; \alpha_{12})$ be an (F_1, F_2)-system and let $\mathcal{B}_{34} = (B; \alpha_{34})$ be an (F_3, F_4)-system and let $\varphi : \mathcal{A}_{12} \to \mathcal{B}_{34}$ be a weak homomorphism.

(1) If $S \subseteq A$ is open in $\mathcal{A}_{12}$, then $\varphi[S]$ is open in $\mathcal{B}_{34}$.

(2) If $R \subseteq B$ is open in $\mathcal{B}_{34}$ and F_2, F_4 preserves pullbacks, then $\varphi^{-1}[R]$ is open in $\mathcal{A}_{12}$.

Proposition 2.5

Let $\mathcal{A}_{12} = (A; \alpha_{12})$, $\mathcal{B}_{34} = (B; \alpha_{34})$ and $\mathcal{C}_{56} = (C; \alpha_{56})$ be an (F_1, F_2)-system, an (F_3, F_4)-system and an (F_5, F_6)-system, respectively and let $\varphi : A \to B, \psi : B \to C$ be mappings.

If (1) F_3 weakly preserves kernels,

(2) F_2, F_4 preserve products and pullbacks,

(3) $\psi \circ \varphi : \mathcal{A}_{12} \to \mathcal{C}_{56}$ is a weak homomorphism,

(4) $\varphi : \mathcal{A}_{12} \twoheadrightarrow \mathcal{B}_{34}$ is a surjective weak homomorphism,

then $\psi : \mathcal{B}_{34} \to \mathcal{C}_{56}$ is a weak homomorphism.

Proposition 2.6

Let $\mathcal{A}_{12} = (A; \alpha_{12})$, $\mathcal{B}_{34} = (B; \alpha_{34})$ and $\mathcal{C}_{56} = (C; \alpha_{56})$ be an (F_1, F_2)-system, an (F_3, F_4)-system and an (F_5, F_6)-system, respectively and let $\varphi : A \to B, \psi : B \to C$ be mappings.

If (1) F_4, F_6 preserve products and pullbacks,

(2) $\psi \circ \varphi : \mathcal{A}_{12} \to \mathcal{C}_{56}$ is a weak homomorphism,

(3) $\psi : \mathcal{B}_{34} \rightarrowtail \mathcal{C}_{56}$ is an injective weak homomorphism,

then $\varphi : \mathcal{A}_{12} \to \mathcal{B}_{34}$ is a weak homomorphism.

Proposition 2.7

Let $\mathcal{A}_{12} = (A; \alpha_{12})$, $\mathcal{B}_{34} = (B; \alpha_{34})$ and $\mathcal{C}_{56} = (C; \alpha_{56})$ be an (F_1, F_2)-system, an (F_3, F_4)-system and an (F_5, F_6)-system, respectively.
If (1) F_3 weakly preserves kernels, and
(2) F_2, F_4 preserve products and pullbacks, and
(3) $\varphi : \mathcal{A}_{12} \twoheadrightarrow \mathcal{B}_{34}$ is a surjective weak homomorphism, and
(4) $\gamma : \mathcal{A}_{12} \to \mathcal{C}_{56}$ is a weak homomorphism,
then there exists a unique weak homomorphism $\psi : \mathcal{B}_{34} \to \mathcal{C}_{56}$ such that $\psi \circ \varphi = \gamma$ iff $Ker\varphi \subseteq Ker\gamma$.

$$\begin{tikzcd}
A \arrow[r, "\gamma"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "(=)"] & C \\
B \arrow[ur, "\exists!\psi"'] &
\end{tikzcd}$$

Definition 2.8

Let $\mathcal{A}$ be an (F_1, F_2)-system. A binary relation $\theta \subseteq A \times A$ is said to be a congruence on $\mathcal{A}$ if there exist an (F_1, F_2)-system $\mathcal{B}$ and a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\theta = Ker\varphi$.

Proposition 2.9

Let $\mathcal{A}_{12} = (A; \alpha_{12})$ be an (F_1, F_2)-system and let $\mathcal{B}_{34} = (B; \alpha_{34})$ be an (F_3, F_4)-system. If $\varphi : \mathcal{A}_{12} \to \mathcal{B}_{34}$ is a weak homomorphism, then $Ker\varphi$ is a congruence on $\mathcal{A}_{12}$.

Proposition 2.10

If $\mathcal{A} = (A; \alpha_A)$ is an (F_1, F_2)-system and θ is a congruence on $\mathcal{A}$, then there is a unique mapping $\alpha_\theta : F_1(A|_\theta) \to F_2(A|_\theta)$ such that the natural mapping $\eta_\theta : A \to A|_\theta$ is a weak homomorphism.

Theorem 2.11

Let $\mathcal{A}_{12} = (A; \alpha_{12})$ be an (F_1, F_2)-system and let $\mathcal{B}_{34} = (B; \alpha_{34})$ be an (F_3, F_4)-system. If $\varphi : \mathcal{A}_{12} \to \mathcal{B}_{34}$ is a surjective weak homomorphism, then $\mathcal{A}_{12}|_{Ker\varphi}$ weak isomorphic to $\mathcal{B}_{34}$.

$$\begin{tikzcd}
A \arrow[r, two heads, "\varphi"] \arrow[d, "\eta_{Ker\varphi}"'] \arrow[dr, phantom, "(=)"] & B \\
A|_{Ker\varphi} \arrow[ur, "\exists!\psi"'] &
\end{tikzcd}$$

References

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THANK YOU FOR YOUR ATTENTION.