## Weak Homomorphisms for $(F_1, F_2)$ -systems

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In 1960, E. Marczewski proposed the concept of weak homomorphisms for non-indexed algebras and in 1980, K.Glazek proposed this concept for indexed algebras.

Let  $\underline{A} = (A; (f_i^{\underline{A}})_{i \in I})$  and  $\underline{B} = (B; (g_j^{\underline{B}})_{j \in J})$  be algebras of types  $\tau_1$ and  $\tau_2$ . A mapping  $\varphi : A \to B$  is said to be a weak homomorphism from  $\underline{A}$  to  $\underline{B}$  if for each  $n_i$ -ary fundamental operation  $f_i^{\underline{A}}$  there exists an  $n_i$ -ary term operation  $s^{\underline{B}}$  of algebra  $\underline{B}$  such that

$$\varphi(f_i^{\underline{A}}(a_1,\ldots,a_{n_i}))=s^{\underline{B}}(\varphi(a_1),\ldots,\varphi(a_{n_i}))$$

for all  $a_1, \ldots, a_{n_i} \in A$  and for each  $n_j$ -ary fundamental operation  $g_j^{\underline{B}}$  there exists an  $n_j$ -ary term operation  $t^{\underline{A}}$  of algebra  $\underline{A}$  such that

$$\varphi(t\underline{^A}(a_1,\ldots,a_{n_j}))=g_j^{\underline{B}}(\varphi(a_1),\ldots,\varphi(a_{n_j}))$$

for all  $a_1, \ldots, a_{n_j} \in A$ .

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Let  $F : Set \to Set$  be a functor. An *F*-algebra  $\mathcal{A}$  is a pair of a set A and a mapping  $\alpha_A : F(A) \to A$ .

Every algebra of type  $\tau$  can be regarded as an *F*-algebra where this functor *F* is called an algebra functor with respect to a type  $\tau$  be defined as follow:

An algebra functor with respect to a type  $\tau = (n_i)_{i \in I}$  is a functor  $F^{\tau} : Set \to Set$  which is defined by:

for each set X , 
$$F^{\tau}(X) = \sum_{i \in I} X^{n_i} = \bigcup_{i \in I} \{(i, \overline{x}) \mid \overline{x} \in X^{n_i}\}$$
  
for each mapping  $\varphi : X \to Y$ ,  $F^{\tau}(\varphi) : \sum_{i \in I} X^{n_i} \to \sum_{i \in I} Y^{n_i}$ 

by 
$$(i, (x_1, \ldots, x_{n_i})) \longmapsto (i, (\varphi(x_1), \ldots, \varphi(x_{n_i}))).$$

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By the definition of algebra functor, we know that

For each algebra 
$$\underline{A} = (A, (f_i^{\underline{A}})_{i \in I})$$
 of type  $\tau$ , there is an  $F^{\tau}$ -algebra  $\mathcal{A} = (A; \alpha_A)$  where  $\alpha_A : F^{\tau}(A) \to A$  by  $\alpha_A(i, (a_1, \ldots, a_{n_i})) = f_i^{\underline{A}}(a_1, \ldots, a_{n_i})$  for all  $(i, (a_1, \ldots, a_{n_i})) \in F^{\tau}(A)$ .

For each 
$$F^{\tau}$$
-algebra  $\mathcal{A} = (A; \alpha_A)$ , there is an algebra  
 $\underline{A} = (A, (f_i^{\underline{A}})_{i \in I})$  of type  $\tau$  where for each  $i \in I$ ,  $f_i^{\underline{A}} : A^{n_i} \to A$  by  
 $f_i^{\underline{A}}(a_1, \ldots, a_{n_i}) = \alpha_A(i, (a_1, \ldots, a_{n_i}))$  for all  $a_1, \ldots, a_{n_i} \in A$ .

Moreover, the category  $Alg\tau$  with objects as algebras and morphisms as homomorphism and the category  $Set^{F\tau}$  with objects as  $F^{\tau}$ -algebra and morphisms as homomorphisms are isomorphic.

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F.M.Schneider generalized concept of K.Glazek to weak homomorphisms for *F*-algebras.

*F*-algebra can be regarded as an  $(F_1, F_2)$ -system where  $F_1 = F$  and  $F_2$  is an identity functor. We will generalize Schneider's idea to weak homomorphisms for  $(F_1, F_2)$ -systems.

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# 1. $(F_1, F_2)$ -Systems

# 2. Weak Homomorphisms for $(F_1, F_2)$ -Systems

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## Definition 1.1

Let  $F_1, F_2 : Set \to Set$  be functors. An  $(F_1, F_2)$ -system  $\mathcal{A}$  is a pair of a set  $\mathcal{A}$  and a mapping  $\alpha_{\mathcal{A}} : F_1(\mathcal{A}) \to F_2(\mathcal{A})$ .

In case  $F_1$  is the identity functor,  $(F_1, F_2)$ -system  $\mathcal{A}$  is said to be an  $F_2$ -coalgebra.

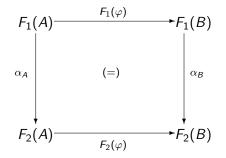
In case  $F_2$  is the identity functor,  $(F_1, F_2)$ -system  $\mathcal{A}$  is said to be an  $F_1$ -algebra.

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#### Definition 1.2

Let  $\mathcal{A} = (A; \alpha_A)$  and  $\mathcal{B} = (B; \alpha_B)$  be  $(F_1, F_2)$ -systems. A mapping  $\varphi : A \to B$  is called a **homomorphism** from  $\mathcal{A}$  to  $\mathcal{B}$ , written as  $\varphi : \mathcal{A} \to \mathcal{B}$ , if

$$F_2(\varphi) \circ \alpha_A = \alpha_B \circ F_1(\varphi).$$



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## Theorem 1.3[2], [3]

Let  $\mathcal{A} = (A; \alpha_A), \mathcal{B} = (B; \alpha_B)$  and  $\mathcal{C} = (C; \alpha_C)$  be  $(F_1, F_2)$ systems and let  $\psi : \mathcal{A} \to \mathcal{B}$  and  $\varphi : \mathcal{B} \to \mathcal{C}$  be homomorphisms.
Then

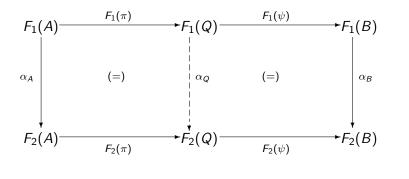
- 1) The identity mapping  $id_A : A \to A$  is a homomorphism from  $\mathcal{A}$  to  $\mathcal{A}$ .
- 2) The composition function  $\varphi \circ \psi : A \to C$  is a homomorphism from  $\mathcal{A}$  to  $\mathcal{C}$ .

The class of all  $(F_1, F_2)$ -systems together with homomorphisms forms a category, written as  $Set_{(F_1,F_2)}$ .

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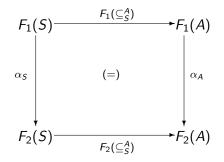
#### Theorem 1.4[3](The Factorization Theorem)

Let  $\mathcal{A} = (A; \alpha_A)$  and  $\mathcal{B} = (B; \alpha_B)$  be  $(F_1, F_2)$ -systems and let  $\varphi : \mathcal{A} \to \mathcal{B}$  be a homomorphism. If  $\varphi = \psi \circ \pi$  is a factorization where  $\pi : \mathcal{A} \twoheadrightarrow \mathcal{Q}$  and  $\psi : \mathcal{Q} \rightarrowtail \mathcal{B}$ , then there is a unique mapping  $\alpha_Q : F_1(\mathcal{Q}) \to F_2(\mathcal{Q})$  such that  $\pi$  and  $\psi$  are homomorphisms.



#### Definition 1.5

Let  $\mathcal{A} = (A; \alpha_A)$  be an  $(F_1, F_2)$ -system. A subset S of A is said to be **open** in  $\mathcal{A}$  if there is a mapping  $\alpha_S : F_1(S) \to F_2(S)$  such that the embedding  $\subseteq_S^A: S \hookrightarrow A$  is a homomorphism, and  $\mathcal{S} = (S; \alpha_S)$ is called an  $(F_1, F_2)$ -subsystem of  $\mathcal{A}$ , written as  $\mathcal{S} \preceq \mathcal{A}$ .

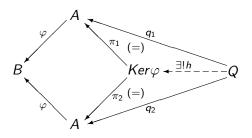


## Proposition 1.6[3]

Let  $\mathcal{A} = (A; \alpha_A)$  and  $\mathcal{B} = (B; \alpha_B)$  be  $(F_1, F_2)$ -systems and let  $\varphi : \mathcal{A} \to \mathcal{B}$  be a homomorphism. (1) If  $S \subseteq A$  is open in  $\mathcal{A}$ , then  $\varphi[S]$  is open in  $\mathcal{B}$ . (2) If  $R \subseteq B$  is open in  $\mathcal{B}$  and  $F_2$  preserves pullbacks, then  $\varphi^{-1}[R]$  is open in  $\mathcal{A}$ .

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On the category Set, we know that for each mapping  $\varphi : A \to B$ , *Ker* $\varphi$  together with canonical projections  $\pi_1, \pi_2 : Ker\varphi \to A$  forms a pullback of  $\varphi$  and  $\varphi$ .



#### Definition 1.7

A functor  $F : Set \to Set$  weakly preserves kernels if for each mapping  $\varphi : A \to B$ ,  $F(Ker\varphi)$  together with  $F(\pi_1), F(\pi_2)$  where  $\pi_1, \pi_2 : Ker\varphi \to A$  are canonical projections, forms a pullback of  $F(\varphi)$  and  $F(\varphi)$ .

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Let  $\mathcal{A} = (A; \alpha_A)$  and  $\mathcal{B} = (B; \alpha_B)$  be  $(F_1, F_2)$ -systems. If  $\varphi$  is a homomorphism from  $\mathcal{A}$  to  $\mathcal{B}$ and  $F_2$  weakly preserves kernels and preserves products, then  $Ker\varphi$  is open in  $\mathcal{A} \times \mathcal{A}$ .

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Let  $\mathcal{A} = (A; \alpha_A)$  be an  $(F_1, F_2)$ -system and let  $\varphi : A \to B$  be a mapping.

If (1)  $\varphi$  is surjective and

(2)  $\mathcal{A} \times \mathcal{A}$  exists and  $Ker\varphi$  is open in  $\mathcal{A} \times \mathcal{A}$  and

(3)  $F_1$  weakly preserves kernels,

then there is a unique mapping  $\alpha_B : F_1(B) \to F_2(B)$  such that  $\varphi$  is a homomorphism from  $\mathcal{A}$  to  $\mathcal{B} = (B; \alpha_B)$ .

#### Proposition 1.10

Let  $\mathcal{B} = (B; \alpha_B)$  be an  $(F_1, F_2)$ -system and let  $\varphi : A \to B$  be a mapping. If (1)  $\varphi$  is injective and (2)  $\varphi[A]$  is open in  $\mathcal{B}$ , then there is a unique mapping  $\alpha_A : F_1(A) \to F_2(A)$  such that  $\varphi$  is a homomorphism from  $\mathcal{A} = (A; \alpha_A)$  to  $\mathcal{B}$ . Let  $\mathcal{A} = (A; \alpha_A)$  be an  $(F_1, F_2)$ -system and let I be an arbitrary set. If  $F_2$  preserves products, then the direct power  $\mathcal{A}^I$  exists and its universe is the direct power  $\mathcal{A}^I$  of the universe of  $\mathcal{A}$  in the category *Set*.

Let  $Sub(\mathcal{A})$  denote the set of all open sets in  $\mathcal{A}$ .

#### Definition 1.11

Let  $\mathcal{A}_{12} = (A; \alpha_{12})$  and  $\mathcal{A}_{34} = (A; \alpha_{34})$  be an  $(F_1, F_2)$ -system and an  $(F_3, F_4)$ -system, respectively. We say that the structural mappings  $\alpha_{12}$  and  $\alpha_{34}$  are **algebraically equivalent**, written as  $\alpha_{12} \equiv \alpha_{34}$ , if  $Sub(\mathcal{A}'_{12}) = Sub(\mathcal{A}'_{34})$  for all set *I*.

Let  $\mathcal{A}_{12} = (A; \alpha_{12})$  and  $\mathcal{B}_{12} = (B; \beta_{12})$  be  $(F_1, F_2)$ -systems, let  $\mathcal{A}_{34} = (A; \alpha_{34})$  and  $\mathcal{B}_{34} = (B; \beta_{34})$  be  $(F_3, F_4)$ -systems, let  $\varphi$  be a homomorphism from  $\mathcal{A}_{12}$  to  $\mathcal{B}_{12}$ and a homomorphism from  $\mathcal{A}_{34}$  to  $\mathcal{B}_{34}$ , and let  $F_2$  and  $F_4$  preserve products and pullbacks. (1) If  $\beta_{12} \equiv \beta_{34}$  and  $\varphi$  is injective, then  $\alpha_{12} \equiv \alpha_{34}$ . (2) If  $\alpha_{12} \equiv \alpha_{34}$  and  $\varphi$  is surjective, then  $\beta_{12} \equiv \beta_{34}$ .

$$\mathcal{A}_{12} = (A; \alpha_{12}) \qquad \qquad \mathcal{B}_{12} = (B; \beta_{12})$$
$$\mathcal{A}_{34} = (A; \alpha_{34}) \qquad \qquad \mathcal{B}_{34} = (B; \beta_{34})$$

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## Definition 2.1

Let  $\mathcal{A}_{12} = (A; \alpha_{12})$  be an  $(F_1, F_2)$ -system and let  $\mathcal{B}_{34} = (B; \beta_{34})$ be an  $(F_3, F_4)$ -system. A mapping  $\varphi : A \to B$  is called a **weak homomorphism** from  $\mathcal{A}_{12}$  to  $\mathcal{B}_{34}$  if for each factorization  $\varphi = \psi \circ \pi$  where  $\pi : A \to Q$  and  $\psi : Q \to B$ , there are mappings  $\gamma_{12} : F_1(Q) \to F_2(Q)$  and  $\gamma_{34} : F_3(Q) \to F_4(Q)$  such that (i)  $\gamma_{12} \equiv \gamma_{34}$ , and (ii)  $\pi$  is a homomorphism from  $\mathcal{A}_{12}$  to  $\mathcal{Q}_{12} = (Q; \gamma_{12})$ , and (iii)  $\psi$  is a homomorphism from  $\mathcal{Q}_{34} = (Q; \gamma_{34})$  to  $\mathcal{B}_{34}$ .

By The Factorization Theorem, every homomorphism is a weak homomorphism. Then identity mapping is a weak homomorphism.

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Let  $A_{12} = (A; \alpha_{12})$  and  $B_{34} = (B; \beta_{34})$  be  $(F_1, F_2)$ -system and  $(F_3, F_4)$ -system, respectively and let  $\varphi : A \to B$  be a mapping. If  $F_2$  and  $F_4$  preserve products and pullbacks, and there is a factorization  $\varphi = \psi \circ \pi$  where  $\pi : A \twoheadrightarrow Q$  and  $\psi : Q \rightarrowtail B$  such that there are mappings  $\gamma_{12} : F_1(Q) \to F_2(Q)$ and  $\gamma_{34} : F_3(Q) \to F_4(Q)$  such that satisfy conditions (i), (ii) and (iii) in definition 2.1,

then  $\varphi$  is a weak homomorphism from  $\mathcal{A}_{12}$  to  $\mathcal{B}_{34}$ .

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#### Theorem 2.3

Let  $\mathcal{A}_{12} = (A; \alpha_{12})$ ,  $\mathcal{B}_{34} = (B; \alpha_{34})$  and  $\mathcal{C}_{56} = (C; \alpha_{56})$  be  $(F_1, F_2)$ -system,  $(F_3, F_4)$ -system and  $(F_5, F_6)$ -system, respectively. If (1)  $F_1$  weakly preserves kernels and (2)  $F_2$ ,  $F_4$  and  $F_6$  preserve products and pullbacks and (3)  $\varphi_1$  is a weak homomorphisms from  $\mathcal{A}_{12}$  to  $\mathcal{B}_{34}$  and (4)  $\varphi_2$  is a weak homomorphisms from  $\mathcal{B}_{34}$  to  $\mathcal{C}_{56}$ , then the composition  $\varphi_2 \circ \varphi_1 : A \to C$  is a weak homomorphism from  $\mathcal{A}_{12}$  to  $\mathcal{C}_{56}$ .

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Let  $\mathcal{K}_1$  be the class of all  $\mathcal{S}\textit{et}\text{-endofunctors}$  which weakly perserve kernels.

Let  $\mathcal{K}_2$  be the class of all  $\mathcal{S}et$ -endofunctors which perserve products and pullbacks.

Then the class of all  $(F_1, F_2)$ -systems where  $F_1 \in \mathcal{K}_1$  and  $F_2 \in \mathcal{K}_2$ , together with weak homomorphisms forms a category, written as  $Set_{(\mathcal{K}_1,\mathcal{K}_2)}$ .

Let  $\mathcal{A}_{12} = (A; \alpha_{12})$  be an  $(F_1, F_2)$ -system and let  $\mathcal{B}_{34} = (B; \alpha_{34})$ be an  $(F_3, F_4)$ -system and let  $\varphi : \mathcal{A}_{12} \to \mathcal{B}_{34}$  be a weak homomorphism. (1) If  $S \subseteq A$  is open in  $\mathcal{A}_{12}$ , then  $\varphi[S]$  is open in  $\mathcal{B}_{34}$ . (2) If  $R \subseteq B$  is open in  $\mathcal{B}_{34}$  and  $F_2, F_4$  preserves pullbacks, then  $\varphi^{-1}[R]$  is open in  $\mathcal{A}_{12}$ .

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Let  $\mathcal{A}_{12} = (A; \alpha_{12})$ ,  $\mathcal{B}_{34} = (B; \alpha_{34})$  and  $\mathcal{C}_{56} = (C; \alpha_{56})$  be an  $(F_1, F_2)$ -system, an  $(F_3, F_4)$ -system and an  $(F_5, F_6)$ -system, respectively and let  $\varphi : A \to B, \psi : B \to C$  be mappings. If (1)  $F_3$  weakly preserves kernels, (2)  $F_2, F_4$  preserve products and pullbacks, (3)  $\psi \circ \varphi : \mathcal{A}_{12} \to \mathcal{C}_{56}$  is a weak homomorphism, (4)  $\varphi : \mathcal{A}_{12} \to \mathcal{B}_{34}$  is a surjective weak homomorphism, then  $\psi : \mathcal{B}_{34} \to \mathcal{C}_{56}$  is a weak homomorphism.

## Proposition 2.6

Let  $\mathcal{A}_{12} = (A; \alpha_{12})$ ,  $\mathcal{B}_{34} = (B; \alpha_{34})$  and  $\mathcal{C}_{56} = (C; \alpha_{56})$  be an  $(F_1, F_2)$ -system, an  $(F_3, F_4)$ -system and an  $(F_5, F_6)$ -system, respectively and let  $\varphi : A \to B, \psi : B \to C$  be mappings. If (1)  $F_4, F_6$  preserve products and pullbacks, (2)  $\psi \circ \varphi : \mathcal{A}_{12} \to \mathcal{C}_{56}$  is a weak homomorphism, (3)  $\psi : \mathcal{B}_{34} \to \mathcal{C}_{56}$  is an injective weak homomorphism, then  $\varphi : \mathcal{A}_{12} \to \mathcal{B}_{34}$  is a weak homomorphism.

Let  $A_{12} = (A; \alpha_{12})$ ,  $B_{34} = (B; \alpha_{34})$  and  $C_{56} = (C; \alpha_{56})$  be an  $(F_1, F_2)$ -system, an  $(F_3, F_4)$ -system and an  $(F_5, F_6)$ -system, respectively.

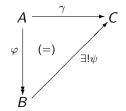
If (1)  $F_3$  weakly preserves kernels, and

(2)  $F_2$ ,  $F_4$  preserve products and pullbacks, and

(3)  $\varphi:\mathcal{A}_{12}\twoheadrightarrow\mathcal{B}_{34}$  is a surjective weak homomorphism, and

(4)  $\gamma : \mathcal{A}_{12} \rightarrow \mathcal{C}_{56}$  is a weak homomorphism,

then there exists a unique weak homomorphism  $\psi : \mathcal{B}_{34} \to \mathcal{C}_{56}$  such that  $\psi \circ \varphi = \gamma$  iff  $Ker \varphi \subseteq Ker \gamma$ .



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## Definition 2.8

Let  $\mathcal{A}$  be an  $(F_1, F_2)$ -system. A binary relation  $\theta \subseteq A \times A$  is said to be a congruence on  $\mathcal{A}$  if there exist an  $(F_1, F_2)$ -system  $\mathcal{B}$  and a homomorphism  $\varphi : \mathcal{A} \to \mathcal{B}$  such that  $\theta = Ker\varphi$ .

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Let  $\mathcal{A}_{12} = (A; \alpha_{12})$  be an  $(F_1, F_2)$ -system and let  $\mathcal{B}_{34} = (B; \alpha_{34})$ be an  $(F_3, F_4)$ -system. If  $\varphi : \mathcal{A}_{12} \to \mathcal{B}_{34}$  is a weak homomorphism, then  $Ker\varphi$  is a congruence on  $\mathcal{A}_{12}$ .

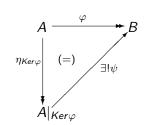
#### Proposition 2.10

If  $\mathcal{A} = (A; \alpha_A)$  is an  $(F_1, F_2)$ -system and  $\theta$  is a congruence on  $\mathcal{A}$ , then there is a unique mapping  $\alpha_{\theta} : F_1(A|_{\theta}) \to F_2(A|_{\theta})$  such that the natural mapping  $\eta_{\theta} : A \to A|_{\theta}$  is a weak homomorphism.

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#### Theorem 2.11

Let  $\mathcal{A}_{12} = (A; \alpha_{12})$  be an  $(F_1, F_2)$ -system and let  $\mathcal{B}_{34} = (B; \alpha_{34})$ be an  $(F_3, F_4)$ -system. If  $\varphi : \mathcal{A}_{12} \to \mathcal{B}_{34}$  is a surjective weak homomorphism, then  $\mathcal{A}_{12}|_{Ker\varphi}$  weak isomorphic to  $\mathcal{B}_{34}$ .



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