Sharply dominating lattice effect algebras and basic decomposition of elements

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Introduction

Lattice effect algebras (introduced by D. Foulis and M.K. Bennett 1994) are a common generalization of orthomodular lattices (hence they may include noncompatible pairs of elements) and MV-algebras (hence they may include unsharp elements):

in the case if every element of an effect algebra is sharp then E is an orthomodular lattice; in the case if every pair of elements of E is compatible then E is an MV-algebra.

Basic definition – effect algebras

Definition (D. Foulis and M.K. Bennett, 1994)

A partial algebra $(E;\oplus,0,1)$ is called an **effect algebra** if 0, 1 are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfy the following conditions for any $x,y,z \in E$:

(Ei) $x\oplus y = y\oplus x$ if $x\oplus y$ is defined,

(Eii) $(x\oplus y)\oplus z = x\oplus(y\oplus z)$ if one side is defined,

(Eiii) for every $x\in E$ there exists a unique $y\in E$ such that $x\oplus y = 1$ (we put $x' = y$),

(Eiv) if $1\oplus x$ is defined then $x = 0$.

On every effect algebra E the partial order $\leq$ and a partial binary operation $\ominus$ can be introduced as follows:

$$x\leq y \text{ and } y\ominus x = z \text{ iff } x\oplus z \text{ is defined and } x\oplus z = y.$$

Basic definitions – effect algebras

If E with the defined partial order is a lattice (a complete lattice) then $(E;\oplus,0,1)$ is called a *lattice effect algebra* (a *complete lattice effect algebra*).

Examples

An orthomodular lattice L becomes a lattice effect algebra if we set $x\oplus y = x\vee y$ for all $x,y\in L$ such that $x\leq y^{\perp}$.

An MV-algebra $M=(M,+,^{*},0,1)$ becomes a lattice effect algebra (called MV-*effect algebra*) if we restrict the total operation $+$ to those elements $x,y\in M$ such that $x\leq y^{*}$, hence $x\oplus y = x+y$ iff $x\leq y^{*}$, for $x,y\in M$.

Basic definitions – effect algebras

Definition

Let E be an effect algebra.

(1) Then $Q \subseteq E$ is called a *sub-effect algebra* of E if
  - (i) $1 \in Q$,
  - (ii) if out of elements $x, y, z \in E$ with $x \oplus y = z$ two are in Q, then $x, y, z \in Q$.

(2) If E is a lattice effect algebra and Q is a sub-lattice and a sub-effect algebra of E then Q is called a *sub-lattice effect algebra* of E.

(3) A sub-lattice effect algebra M of a lattice effect algebra E is called a *block* of E if
  - (i) $x \leftrightarrow y$ (*compatible*, i.e., $x \vee y = x \oplus (y \ominus (x \wedge y))$) for all $x, y \in M$,
  - (ii) if $z \in E$ and $x \leftrightarrow z$ for all $x \in M$ then $z \in M$.

Theorem

(Z.R., 2000) Every maximal subset of pairwise compatible elements of a lattice effect algebra E is a block of E and $E = \bigcup\{M \subseteq E \mid M$ block of $E\}$. Moreover, every block of E is a sub-MV-effect algebra of E.

Archimedean atomic lattice effect algebras

Definition

A lattice effect algebra E is

(1) *Archimedean* if for all $x \in E$, $x \neq 0$ there exists positive integer $n_x = max\{n \in \mathbb{N} \mid nx = \underbrace{x \oplus x \oplus \cdots \oplus x}_{n\text{-times}} \text{ exists}\}$.

(2) *atomic* if under every nonzero element of E there is an atom (*minimal nonzero element*).

Remark

If an Archimedean atomic lattice effect algebra E is not an orthomodular lattice then E need not be atomistic.

Example

The finite chain MV-effect algebra $E = \{0, a, 2a, \ldots, n_a a\}$ is not atomistic, i.e., there exists $x \in E$ with $x \neq \bigvee\{a \in E \mid a \leq x, a \text{ an atom}\}$.

Archimedean atomic lattice effect algebras

Theorem

Let E be an Archimedean atomic lattice effect algebra. Then

(i) *(Z.R. 2002) For every nonzero element $x \in E$ there are mutually distinct atoms $a_\alpha \in E$, $\alpha \in \mathscr{E}$ and positive integers k_α, $\alpha \in \mathscr{E}$ such that*

$$x = \bigoplus \{k_\alpha a_\alpha \mid \alpha \in \mathscr{E}\} = \bigvee \{k_\alpha a_\alpha \mid \alpha \in \mathscr{E}\}$$

under which $x \in S(E)$ iff $k_\alpha = n_{a_\alpha} = \mathrm{ord}(a_\alpha)$ for all $\alpha \in \mathscr{E}$.

(ii) *If E is an MV-effect algebra (equivalently E has a unique block) then the decomposition in (i) is unique, i.e., the set $\{a_\alpha \in E \mid \alpha \in \mathscr{E}\}$ and positive integers k_α, $\alpha \in \mathscr{E}$ are unique.*

(iii) *If E is not an MV-effect algebra (equivalently E has more than one block) then the decomposition in (i) need not be unique ($1 = a \oplus b = 2c$, a, b, c are atoms).*

Sharp elements of lattice effect algebras

Definition

(S. Gudder, 1998) Let E be an effect algebra. An element $x \in E$ is *sharp* iff $x \wedge x' = 0$. Set $S(E) = \{x \in E \mid x \wedge x' = 0\}$.

Theorem

(i) *(G. Jenča and Z. R. 1999) For every lattice effect algebra E the set $S(E)$ is a sub-effect algebra and a full sub-lattice of E (i.e., ($D \subseteq S(E)$ and $\bigvee_E D$ exists) implies ($\bigvee_{S(E)} D$ exists and $\bigvee_E D = \bigvee_{S(E)} D$)) and $(S(E), \vee, \wedge, ', 0, 1)$ is an orthomodular lattice.*

(ii) *(J. Paseka and Z. R. 2009) If E is an Archimedean atomic lattice effect algebra then $S(E)$ is a bifull sub-lattice of E (i.e., for any $D \subseteq S(E)$, $\bigvee_E D$ exists iff $\bigvee_{S(E)} D$, in which case $\bigvee_E D = \bigvee_{S(E)} D$).*

(iii) *If E is a complete lattice effect algebra then $S(E)$ is a complete sub-lattice (hence a bifull sub-lattice) of E.*

Sharply dominating lattice effect algebras

Definition

(S. Gudder 1998) An effect algebra $(E, \oplus, 0, 1)$ is called *sharply dominating* if for every $x \in E$ there exists a smallest sharp element w^x such that $x \leq w^x \in S(E)$ (i.e., $w \in S(E)$ satisfies $x \leq w$ then $w^x \leq w$).

Example

Clearly, every complete lattice effect algebra E is sharply dominating.

Proposition

If $S(E)$ of an Archimedean atomic lattice effect algebra E is a complete orthomodular lattice then E is sharply dominating (since $S(E)$ is a bifull sub-lattice of E).

Remark (Archimedean atomic lattice effect algebra E is sharply dominating) $\not\Rightarrow$ $S(E)$ is a complete orthomodular lattice; e.g. if E is a Boolean algebra which is not a complete lattice.

Basic decomposition of elements (BDE-property) of Archimedean atomic lattice effect algebras

Definition

Let E be an Archimedean lattice effect algebra.

1. $x \in E$ *has a BDE-property* if there are the unique $w_x \in S(E)$, unique set of atoms $\{a_\alpha | \alpha \in \Lambda\}$ of E and unique positive integers $k_\alpha < n_{a_\alpha}$ such that
$$x = w_x \oplus \left(\bigoplus\{k_\alpha a_\alpha | \alpha \in \Lambda\}\right).$$
2. E *has a BDE-property* if every element $x \in E$ has a BDE-property.

Theorem

(Z. R. and Wu Junde 2008) An Archimedean atomic lattice effect algebra E has the BDE-property iff E is sharply dominating.

Basic decomposition of elements (BDE-property) of Archimedean atomic lattice effect algebras

Remark

1. *In the BDE of $x \in E$, the unique $w_x \in S(E)$ is in fact the greatest sharp element under x, hence $e = x \ominus w_x$ is a* meager element *(i.e., $(w \in S(E), w \leq e) \implies w = 0$) and the decomposition $x = w_x \oplus e$ is unique.*
2. *M. Kalina, V. Olejček (preprint 2009) showed that there are even Archimedean atomic MV-effect algebras which are not sharply dominating, i.e., in the decomposition $x = \bigoplus\{k_\alpha a_\alpha | \alpha \in \Lambda\}$ the elements $\bigoplus\{k_\alpha a_\alpha | \alpha \in \Lambda, k_\alpha = n_{a_\alpha}\}$ and $\bigoplus\{k_\alpha a_\alpha | \alpha \in \Lambda, k_\alpha < n_{a_\alpha}\}$ need not exist.*

Blocks of lattice effect algebras

Theorem

(K. Mosná, 2007) Every Archimedean atomic lattice effect algebra E is a union of its atomic blocks. Every atomic block M uniquely corresponds to a maximal pairwise compatible set A_M of atoms of E.

Theorem

Let E be a lattice effect algebra and M be an Archimedean atomic block of E. Then

1. *M is a bifull sub-lattice of E.*
2. *If $A_M = \{a_\kappa \mid \kappa \in H, a_\kappa \text{ atom of } M\}$ is a maximal pairwise compatible set of atoms then*

$$\bigoplus_E \{n_{a_\kappa} a_\kappa | \kappa \in H\} = \bigoplus_M \{n_{a_\kappa} a_\kappa | \kappa \in H\} = 1.$$

Blocks of sharply dominating Archimedean atomic lattice effect algebras

Theorem

Let E be a sharply dominating Archimedean atomic lattice effect algebra. Then

1. *Every atomic block M of E is sharply dominating.*
2. *Every atomic block M of E is a bifull sub-lattice of E (i.e., for any $D \subseteq M$, $\bigvee_E D$ exists iff $\bigvee_M D$ exists, in which case $\bigvee_E D = \bigvee_M D$).*
3. *For every $x \in E$, the BDE of x in E coincides with BDE of x in every atomic block M of E including x.*

Blocks of of sharply dominating Archimedean atomic lattice effect algebras

Theorem

Let E be an Archimedean atomic lattice effect algebra. The following conditions are equivalent:

(i) *Every atomic block M of E is sharply dominating.*

(ii) *For every atomic block M of E and every $x \in M$, x has the BDE-property both in M and E, i.e., the BDE of x in E coincides with the BDE of x in M.*

(iii) *E is sharply dominating.*

The center $C(E)$ and the center of compatibility $B(E)$ of sharply dominating Archimedean atomic lattice effect algebra

Definition

Let E be a lattice effect algebra. $B(E) = \bigcap\{M \subseteq E \mid M$ is a block of $E\} = \{x \in E \mid x \leftrightarrow y$ for every $y \in E\}$ is called a *center of compatibility* of E. $C(E) = B(E) \cap S(E) = \{x \in E \mid y = (y \wedge x) \vee (y \wedge x')$ for all $y \in E\}$ is called a *center* of E.

Theorem

Let E be a sharply dominating Archimedean atomic lattice effect algebra. Then

(i) *$B(E)$ is a sharply dominating MV-effect algebra.*

(ii) *For every $x \in B(E), x \neq 0$ there exist unique $w_x \in C(E)$, unique set $\{a_\alpha | \alpha \in \Lambda\} \subseteq B(E)$ of atoms of E and unique positive integers $k_\alpha < n_{a_\alpha}$ such that $x = w_x \oplus (\bigoplus\{k_\alpha a_\alpha | \alpha \in \Lambda\})$.*

(iii) *If $C(E) = \{0, 1\}$ then either $B(E) = C(E)$ or $E = B(E) = \{0, a, 2a, \ldots, 1 = n_a a\}$.*

The center $C(E)$ and the center of compatibility $B(E)$ of sharply dominating Archimedean atomic lattice effect algebra

Theorem

Let E be a lattice effect algebra. Let at least one block M of E be complete and atomic. Then

(i) *$C(E)$ and $B(E)$ are complete and atomic.*

(ii) *$C(E)$ and $B(E)$ are bifull sub-lattices of both E and M.*

Applications

(A) *In questions on the existence of states (or probability measures) on a lattice effect algebra* $E \neq S(E)$:
For every sharply dominating Archimedean atomic lattice effect algebra E (equivalently with BDE-property) the existence of an (o)-continuous state on E implies the existence of a state on $S(E)$ (Z.R., Wu Junde, 2008). Consequently, if E is a finite lattice effect algebra then a state on E exists iff a state on $S(E)$ exists. Moreover, if $B(E) \neq C(E)$ then there exists an extremal (o)-continuous state ω on E, which is subadditive.

(B) *In questions for a lattice effect algebras to be a sub-direct product of irreducible ones*:
If the center $C(E)$ of a lattice effect algebra E is atomic and bifull sub-lattice of E then E is isomorphic to a subdirect product of $\prod\{[0,p] \mid p \in E$ is an atom of $C(E)\}$, where intervals $[0,p]$ with inherited $\oplus$-operation are *irreducible* lattice effect algebras, meaning that $C([0,p) = \{0,p\}$ (Z. R., 2003).

Thank you for your attention.