Generalized fuzzy topology versus non-commutative topology

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Outline

Fixed-basis fuzzy topology

Motivating idea

Develop the mathematics of fuzzy or cloudy quantities which are not described in terms of probability distributions.

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1968 C. L. Chang introduces fixed-basis fuzzy topology as a subset of the powerset $[0,1]^X$ closed under arbitrary $\bigvee$ and finite $\wedge$.

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Different topological spaces may have different lattices serving as a basis for the respective powerset.

1980 B. Hutton uses fuzzy lattice (completely distributive lattice with an order reversing involution) to obtain a variable-basis category of singleton topological spaces.

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Remove the requirement of commutativity in the Gelfand-Neumark duality between the categories of Hausdorff locally compact topological spaces and commutative C^*-algebras.

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1989 F. Borceux and G. van den Bossche introduce quantum space making the usual frame of open sets into a right-sided idempotent quantale.

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- variable-basis approach to fuzzy topology of S. E. Rodabaugh;
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!!! The link between two concepts is given by generalized fuzzy sets of N. Nakajima defined as points of a product $\prod_{x \in X} L_x$ of complete lattices, every $x \in X$ having its own lattice of membership degrees.

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Ω-algebras and Ω-homomorphisms

- Suppose $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ is a class of cardinal numbers.

Definition 1

- An Ω-algebra is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$ consisting of a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$, called n_λ-ary operations on A.
- An Ω-homomorphism $(A, (\omega_\lambda^A)_{\lambda \in \Lambda}) \xrightarrow{f} (B, (\omega_\lambda^B)_{\lambda \in \Lambda})$ is a map $A \xrightarrow{f} B$ such that $f \circ \omega_\lambda^A = \omega_\lambda^B \circ f_\lambda^n$ for every $\lambda \in \Lambda$.
- $\mathbf{Alg}(\Omega)$ is the category of Ω-algebras and Ω-homomorphisms, the underlying functor to the ground category $\mathbf{Set}$ of sets and maps denoted by $|-|$.

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Algebras and homomorphisms

- Suppose $\mathcal{M}$ (resp. $\mathcal{E}$) is the class of Ω-homomorphisms with injective (resp. surjective) underlying maps.

Definition 2

- A variety of Ω-algebras is a full subcategory of $\mathbf{Alg}(\Omega)$ closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras) and $\mathcal{E}$-quotients (homomorphic images).
- The objects (resp. morphisms) of a variety will be referred to as algebras (resp. homomorphisms).

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Localic algebras

- From now on assume that **A** is a fixed variety.

Definition 3

The dual of the category **A** is denoted by **LoA** (the "**Lo**" comes from "localic"). Its objects (resp. morphisms) are called localic algebras (resp. homomorphisms).

!!! Given a morphism f of a category **C**, the respective morphism of $\mathbf{C}^{op}$ is denoted by f^{op} and vice versa.

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Ground category for generalized topology

Definition 4

- **Set** $\odot$ **LoA** is the category, the objects of which are pairs $(X, \mathcal{A})$, where X is a set and $\mathcal{A} = (A_x)_{x \in X}$ is a family of localic algebras.
- Morphisms $(X, \mathcal{A}) \xrightarrow{(f,\Phi)} (Y, \mathcal{B})$ consist of a map $X \xrightarrow{f} Y$ and a family $\Phi = (\varphi_x)_{x \in X}$ of localic homomorphisms $A_x \xrightarrow{\varphi_x} B_{f(x)}$.
- The composition of two morphisms $(X, \mathcal{A}) \xrightarrow{(f,\Phi)} (Y, \mathcal{B})$ and $(Y, \mathcal{B}) \xrightarrow{(g,\Psi)} (Z, \mathcal{C})$ is given by $(g, \Psi) \circ (f, \Phi) = (g \circ f, \Psi \circ \Phi)$, where $\Psi \circ \Phi = (\psi_{f(x)} \circ \varphi_x)_{x \in X}$.
- The identity on $(X, \mathcal{A})$ is given by $(X, \mathcal{A}) \xrightarrow{(1_X, 1_{\mathcal{A}})} (X, \mathcal{A})$, where $1_{\mathcal{A}} = (1_{A_x})_{x \in X}$.

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The nature of the ground category

Theorem 5

Set $\odot$ **LoA** *is a free coproduct completion of* **LoA***, namely:*

- *there exists a full embedding* $\mathbf{LoA} \overset{E}{\hookrightarrow} \mathbf{Set} \odot \mathbf{LoA}$;
- $\mathbf{Set} \odot \mathbf{LoA}$ *has coproducts;*
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There exists a non-full embedding $\mathbf{Set} \times \mathbf{LoA} \overset{E}{\hookrightarrow} \mathbf{Set} \odot \mathbf{LoA}$,

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Generalized powerset operator

Given a family of localic algebras $(A_x)_{x \in X}$, denote their product $\prod_{x \in X} A_x$ by $\mathcal{A}^X$ and consider it as the set of choice functions on X, i.e., maps $X \xrightarrow{p} \bigcup_{x \in X} A_x$ such that $p(x) \in A_x$.

Lemma 7

There exists a functor $\mathbf{Set} \odot \mathbf{LoA} \xrightarrow{(-)^{\leftarrow}} \mathbf{LoA}$, *defined by*

$$((X, \mathcal{A}) \xrightarrow{(f,\Phi)} (Y, \mathcal{B}))^{\leftarrow} = \mathcal{A}^X \xrightarrow{((f,\Phi)^{\leftarrow})^{op}} \mathcal{B}^Y,$$

where

$$((f,\Phi)^{\leftarrow}(p))(x) = (\Phi^{op} \circ p \circ f)(x) = (\varphi^{op}_{(-)} \circ p \circ f)(x) = \varphi^{op}_x \circ p \circ f(x).$$

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From generalized to standard

Lemma 8

The composition $\mathbf{Set} \times \mathbf{LoA} \xhookrightarrow{E} \mathbf{Set} \odot \mathbf{LoA} \xrightarrow{(-)^{\leftarrow}} \mathbf{LoA}$ *gives the powerset operator* $\mathbf{Set} \times \mathbf{LoA} \xrightarrow{(-)^{\leftarrow}} \mathbf{LoA}$ *used for the usual variety-based topology and defined by*

$$((X,A) \xrightarrow{(f,\varphi)} (Y,B))^{\leftarrow} = A^X \xrightarrow{((f,\varphi)^{\leftarrow})^{op}} B^Y,$$

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$$(f,\varphi)^{\leftarrow}(p) = \varphi^{op} \circ p \circ f.$$

Generalized topology

Definition 9

Given a subcategory **C** of **LoA** and a **Set** $\odot$ **C**-object $(X, \mathcal{A})$, a subset τ of $\mathcal{A}^X$ is called a generalized **C**-topology on $(X, \mathcal{A})$ provided that τ is a subalgebra of $\mathcal{A}^X$.

Example 10

Suppose X is a set, Q is a s(emi)-quantale and A is an algebra.

- The usual topology on X is a subframe of the powerset $\mathcal{P}(X)$.
- A Q-topology on X is a sub(s-quantale) of the powerset Q^X.
- An A-topology on X is a subalgebra of the powerset A^X.

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Generalized topology

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Given a subcategory **C** of **LoA** and a **Set** $\odot$ **C**-object $(X, \mathcal{A})$, a subset τ of $\mathcal{A}^X$ is called a generalized **C**-topology on $(X, \mathcal{A})$ provided that τ is a subalgebra of $\mathcal{A}^X$.

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$\boxed{!!!}$ Every map $X \xrightarrow{f} Y$ gives the image operator $\mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$.

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- A generalized **C**-topological space is a triple $(X, \mathcal{A}, \tau)$, where τ is a generalized **C**-topology on $(X, \mathcal{A})$.
- A generalized **C**-continuous map $(X, \mathcal{A}, \tau) \xrightarrow{(f,\Phi)} (Y, \mathcal{B}, \sigma)$ is a $\mathbf{Set} \odot \mathbf{C}$-morphism $(X, \mathcal{A}) \xrightarrow{(f,\Phi)} (Y, \mathcal{B})$ with $((f,\Phi)^{\leftarrow})^{\rightarrow}(\sigma) \subseteq \tau$.
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- Given a generalized **C**-topological space $(X, \mathcal{A}, \tau)$, $\mathcal{A}$ is called the basis of $(X, \mathcal{A}, \tau)$.
- Given $x \in X$, A_x is called the x-basis of $(X, \mathcal{A}, \tau)$.
- **C-Top** is the non-full subcategory of **C-GTop** comprising all spaces $(X, (A)_{x \in X}, \tau)$ and all continuous maps $(f, (\varphi)_{x \in X})$.

Example 13

For a subcategory **C** of **LoSQuant**, **C-Top** gives the category of variable-basis topological spaces of S. E. Rodabaugh and **C-GTop** is the category of generalized topological spaces of M. Demirci.

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Motivating example

Definition 14 (C. J. Mulvey and J. W. Pelletier)

- A quantal space $(Q,\mathcal{T}_Q)$ is a Gelfand quantale Q together with an algebraically strong right embedding $Q \xrightarrow{\mathcal{T}_Q} \prod_{i\in I} Q_i$ into a product $\prod_{i\in I} Q_i$ of discrete Hilbert quantales, called the quantal topology.
- A homomorphism $(Q,\mathcal{T}_Q) \xrightarrow{(\varphi,\mathcal{T}_\varphi)} (P,\mathcal{T}_P)$ of quantal spaces is a Gelfand quantale homomorphism $Q \xrightarrow{\varphi} P$ and a discrete von Neumann quantale homomorphism $\prod_{i\in I} Q_i \xrightarrow{\mathcal{T}_\varphi} \prod_{j\in J} P_j$ such that the following diagram commutes

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Algebral spaces

Definition 15

- Given a **Set** $\odot$ **LoA**-object $(X, \mathcal{A})$ and an algebra A, an algebral topology on $(A, (X, \mathcal{A}))$ is a homomorphism $A \xrightarrow{\mathcal{T}} \mathcal{A}^X$.
- An algebral space is a tuple $(A, (X, \mathcal{A}), \mathcal{T})$, where $\mathcal{T}$ is an algebral topology on $(A, (X, \mathcal{A}))$.

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Category of algebral spaces

- Suppose **C** (resp. **D**) is a subcategory of **A** (resp. **LoA**).

Definition 16

- $(\mathbf{C}, \mathbf{D})$-**AlgSp** is the category, the objects of which are $(\mathbf{C}, \mathbf{D})$-algebral spaces, i.e., algebral spaces $(A, (X, \mathcal{A}), \mathcal{T})$ with A in **C** and $\mathcal{A}$ in **D**.
- Morphisms $(A, (X, \mathcal{A}), \mathcal{T}) \xrightarrow{(\varphi,(f,\Phi)^{op})} (B, (Y, \mathcal{B}), \mathcal{S})$ are $\mathbf{C} \times (\mathbf{Set} \odot \mathbf{D})^{op}$-morphisms $(A, (X, \mathcal{A})) \xrightarrow{(\varphi,(f,\Phi)^{op})} (B, (Y, \mathcal{B}))$ making the following diagram commute
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- The underlying functor to the category $\mathbf{C} \times (\mathbf{Set} \odot \mathbf{D})^{op}$ is $|-|$.

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From fuzzy to non-commutative

Lemma 17

There is a full embedding $\mathbf{LoA}\text{-}\mathbf{GTop} \overset{\mathsf{E}}{\hookrightarrow} ((\mathbf{A}, \mathbf{LoA})\text{-}\mathbf{AlgSp})^{op}$ *defined by*

$$\mathsf{E}((X, \mathcal{A}, \tau) \xrightarrow{(f,\Phi)} (Y, \mathcal{B}, \sigma)) =$$

$$(\tau, (X, \mathcal{A}), \iota_\tau) \xrightarrow{((f,\Phi)^{\leftarrow}, (f,\Phi)^{op})^{op}} (\sigma, (Y, \mathcal{B}), \iota_\sigma),$$

where ι_τ *(resp.* ι_σ*) are the inclusion maps.*

From non-commutative to fuzzy

Lemma 18

There is a functor $((\mathbf{A}, \mathbf{LoA})\text{-}\mathbf{AlgSp})^{op} \xrightarrow{\mathrm{Spat}} \mathbf{LoA}\text{-}\mathbf{GTop}$ *defined by*

$$\mathrm{Spat}((A,(X,\mathcal{A}),\mathcal{T}) \xrightarrow{(\varphi,(f,\Phi)^{op})^{op}} (B,(Y,\mathcal{B}),\mathcal{S})) =$$
$$(X,\mathcal{A},\mathcal{T}^{\rightarrow}(A)) \xrightarrow{(f,\Phi)} (Y,\mathcal{B},\mathcal{S}^{\rightarrow}(B)).$$

The main result

Theorem 19

Spat *is a right-adjoint-left-inverse to* E.

Corollary 20

$\mathbf{LoA}$-$\mathbf{GTop}$ *is isomorphic to a full coreflective subcategory of* $((\mathbf{A}, \mathbf{LoA})$-$\mathbf{AlgSp})^{op}$.

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Fuzzy versus non-commutative

Consequences

1. Corollary 20 shows that the non-commutative approach gives a more general framework for developing topology than the respective fuzzy one does.
2. Quantal spaces of C. J. Mulvey and J. W. Pelletier provide a generalization of fuzzy topology developed in the framework of quantales.

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Topological systems

Definition 21

- Let **C** be a subcategory of **LoA**. A **C**-topological system is a tuple $D = (\mathrm{pt}\, D, \Sigma\, D, \Omega\, D, \kappa)$, where $(\mathrm{pt}\, D, \Sigma\, D, \Omega\, D)$ is a **Set** $\times$ **C** $\times$ **C**-object and $\Omega\, D \xrightarrow{\kappa} (\Sigma\, D)^{\mathrm{pt}\, D}$ is a homomorphism.
- A **C**-continuous map $D_1 \xrightarrow{f} D_2$ is a **Set** $\times$ **C** $\times$ **C**-morphism $(\mathrm{pt}\, D_1, \Sigma\, D_1, \Omega\, D_1) \xrightarrow{f=(\mathrm{pt}\, f, (\Sigma f)^{op}, (\Omega f)^{op})} (\mathrm{pt}\, D_2, \Sigma\, D_2, \Omega\, D_2)$ making the following diagram commute

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- The category **C-TopSys** comprises **C**-topological systems and **C**-continuous maps, with the underlying functor to the ground category **Set** $\times$ **C** $\times$ **C** denoted by $|-|$.

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!!! The category $(\mathbf{C}, \mathbf{D})$-**AlgSp** of algebral spaces generalizes the category **C**-**TopSys** of topological systems.

Problem 22

Investigate algebral spaces from the point of view of topological systems, trying to provide analogues of the already existing results for the latter structures.

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- By Lemma 8 the composition

$$\mathbf{Set} \times \mathbf{LoA} \xhookrightarrow{\mathsf{E}} \mathbf{Set} \odot \mathbf{LoA} \xrightarrow{(-)^{\leftarrow}} \mathbf{LoA}$$

gives the standard powerset operator $\mathbf{Set} \times \mathbf{LoA} \xrightarrow{(-)^{\leftarrow}} \mathbf{LoA}$.

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Which functors $\mathbf{Set} \times \mathbf{LoA} \xrightarrow{F} \mathbf{C}$ *are extendable to* $\mathbf{Set} \odot \mathbf{LoA} \xrightarrow{\overline{F}} \mathbf{C}$*?*

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Thank you for your attention!