Operators on varieties of monoids related to polynomial operators on classes of regular languages

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Polynomial operator

The polynomial operator assigns to each class of languages $\mathscr{V}$ the class of all (positive) boolean combinations of the languages of the form

$$L_0 a_1 L_1 a_2 \ldots a_\ell L_\ell \ , \qquad (*)$$

where A is an alphabet, $a_1, \ldots, a_\ell \in A$, $L_0, \ldots, L_\ell \in \mathscr{V}(A)$ (i.e. they are over A).

The resulting classes are denoted by $\mathsf{PPol}\mathscr{V}$ and $\mathsf{BPol}\mathscr{V}$, respectively.

In the restricted case we fix a natural number k and we allow only $\ell \leq k$ in $(*)$. We get the classes $\mathsf{PPol}_k\mathscr{V}$ and $\mathsf{BPol}_k\mathscr{V}$, respectively.

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Basic examples

1. Let $\mathscr{T}(A) = \{\emptyset, A^*\}$ for each finite set A. Then PPol$\mathscr{T}$ is level 1/2 of the Straubing-Thérien hierarchy and BPol$\mathscr{T} = \mathscr{V}_1$ is level 1, i.e. the piecewise testable languages.
Result (Simon - 1972): Decidability of the membership problem for the class $\mathscr{V}_1$.
Open problem: Decidability of the membership problem for the class BPol$\mathscr{V}_1 = \mathscr{V}_2$.

2. Let $\mathscr{S}^+(A)$ be the set of all finite unions of the languages of the form B^*, where $B \subseteq A$, for each finite set A.
Result (Pin, Straubing): BPol$\mathscr{S}^+ = \mathscr{V}_2$.
Open problem – reformulation:
Is it decidable whether a given regular language $L \subseteq A^*$ can be expressed as a boolean combination languages of the form $B_0^* a_1 B_1^* a_2 \dots a_\ell B_\ell^*$, where $a_1, \dots, a_\ell \in A$, $B_0, \dots, B_\ell \subseteq A$.

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Other examples

3. Let $\mathscr{S}(A)$ be the set of all finite unions of the languages of the form $\overline{B}$, where $B \subseteq A$, for each finite set A. Here $\overline{B}$ is the set of all words over A containing exactly the letters from B.

4. Let m be a fixed natural number. Let $\mathscr{A}_m(A)$ be the set of all boolean combinations of the languages of the form $L(a,r) = \{ u \in A^* \mid |u|_a \equiv r \pmod{m} \}$, where $a \in A$ and $0 \leq r < m$, for each finite set A.

Notice that the classes $\mathscr{T}$, $\mathscr{S}$, $\mathscr{A}_m$ are boolean varieties and $\mathscr{S}^+$ is a positive variety.

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Varieties of languages

A boolean variety of languages $\mathscr{V}$ associates to every finite alphabet A a class $\mathscr{V}(A)$ of regular languages over A in such a way that

- $\mathscr{V}(A)$ is closed under finite unions, finite intersections and complements (in particular, $\emptyset, A^* \in \mathscr{V}(A)$),
- $\mathscr{V}(A)$ is closed under derivatives, i.e.
  $L \in \mathscr{V}(A),\ u, v \in A^*$ implies
  $u^{-1}Lv^{-1} = \{ w \in A^* \mid uwv \in L \} \in \mathscr{V}(A)$,
- $\mathscr{V}$ is closed under inverse morphisms, i.e.
  $f : B^* \to A^*,\ L \in \mathscr{V}(A)$ implies
  $f^{-1}(L) = \{ v \in B^* \mid f(v) \in L \} \in \mathscr{V}(B)$.

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Pseudovarieties and varieties of monoids

A pseudovariety of finite (ordered) monoids is a class of finite monoids closed under submonoids, morphic images and products of finite families. Similarly for ordered monoids. When defining a variety of (ordered) monoids we use arbitrary products.

The pseudovarieties of ordered monoids can be characterized by pseudoidentities. The pseudovarieties we consider here are equational – they are given by identities, or equivalently, they are of the form $\mathrm{Fin}\,\mathbf{V}$ where $\mathbf{V}$ is a variety of (ordered) monoids.

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Syntactic structures

For a regular language $L \subseteq A^*$, we define the relations $\sim_L$ and $\preceq_L$ on A^* as follows: for $u, v \in A^*$ we have

$$u \sim_L v \text{ if and only if } (\ \forall\, p, q \in A^*\ )\ (\ puq \in L \iff pvq \in L\ ),$$

$$u \preceq_L v \text{ if and only if } (\ \forall\, p, q \in A^*\ )\ (\ pvq \in L \implies puq \in L\ ).$$

The relation $\sim_L$ is the syntactic congruence of L on A^*. It is of finite index (i.e. there are finitely many classes), the quotient structure $\mathrm{M}(L) = A^*/\sim_L$ is called the syntactic monoid of L.

The relation $\preceq_L$ is the syntactic quasiorder of L and we have $\preceq_L \cap \succeq_L = \sim_L$. Hence $\preceq_L$ induces an order on $\mathrm{M}(L) = A^*/\sim_L$, namely: $u{\sim_L} \leq v{\sim_L}$ if and only if $u \preceq_L v$. We speak about the syntactic ordered monoid of L; we denote the structure by $\mathrm{O}(L)$.

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Eilenberg-type theorems

Result (Eilenberg, Pin)

Boolean varieties (positive varieties) of languages correspond to pseudovarieties of finite monoids (ordered monoids) . The correspondence, written $\mathscr{V} \longleftrightarrow \mathbf{V}$ *(*$\mathscr{P} \longleftrightarrow \mathbf{P}$*), is given by the following relationship: for* $L \subseteq A^*$ *we have*

$$L \in \mathscr{V}(A) \textit{ if and only if } \mathrm{M}(L) \in \mathbf{V}$$

$$(\ L \in \mathscr{P}(A) \textit{ if and only if } \mathrm{O}(L) \in \mathbf{P}\ ).$$

Examples

Pseudovarieties of (ordered) monoids corresponding to the classes $\mathscr{T}, \mathscr{S}^+, \mathscr{S}, \mathscr{A}_m$ consist exactly of all finite members of the following varieties:

$$\mathbf{T} = \mathrm{Mod}(\, x = y \,),\ \mathbf{S}^+ = \mathrm{Mod}(\, x^2 = x,\ xy = yx, 1 \leq x \,),$$

$$\mathbf{S} = \mathrm{Mod}(\, x^2 = x,\ xy = yx \,),\ \mathbf{A}_m = \mathrm{Mod}(\, xy = yx,\ x^m = 1 \,)\ .$$

The names for the (ordered) monoids of the pseudovarieties $\mathbf{T}$, $\mathbf{S}^+$, $\mathbf{S}$, $\mathbf{A}_m$ are trivial monoids (semilattices with the smallest element 1, semilattices and abelian groups of index m, respectively)

Finite characteristics

Let $X = \{x_1, x_2, \dots\}$. A relation γ on X^* is a finite characteristic if it satisfies the following conditions:
(i) γ is a quasiorder on X^*;
(ii) γ is compatible with the multiplication, i.e. for each $u, v, w \in X^*$ we have

$$u \,\gamma\, v \quad \text{implies} \quad uw \,\gamma\, vw, \quad wu \,\gamma\, wv \;;$$

(iii) γ is fully invariant, i.e. for each morphism $\varphi : X^* \to X^*$ and each $u, v \in X^*$ we have

$$u \,\gamma\, v \quad \text{implies} \quad \varphi(u) \,\gamma\, \varphi(v) \;;$$

(iv) for each finite subset Y of the set X, the set Y^* intersects only finitely many classes of $X^*/\gamma \cap \gamma^{-1}$.

Proposition

Positive varieties of languages having all $\mathscr{V}(A)$ finite correspond to finite characteristics. Namely $\mathscr{V} \mapsto \mathrm{Id}\,\mathbf{V}$ and
$L \in \mathscr{V}(A) \quad$ *iff* $\quad \gamma \mid A^* \times A^* \subseteq \preceq_L$.

The classes of languages in our basic examples have the following finite characteristics:

1. $\mathrm{Id}\,\mathbf{T} = X^* \times X^*$.
2. $\mathrm{Id}\,\mathbf{S}^+ = \{ (u, v) \in X^* \times X^* \mid \mathrm{c}(u) \subseteq \mathrm{c}(v) \}$.
3. $\mathrm{Id}\,\mathbf{S} = \{ (u, v) \in X^* \times X^* \mid \mathrm{c}(u) = \mathrm{c}(v) \}$.
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Factorizations

Let k be a fixed natural number and γ be a finite characteristic. For a word $u \in X^*$, we say that

$$f = (u_0, a_1, u_1, a_2, \ldots, a_\ell, u_\ell)$$

is a factorization of u of length ℓ if $u_0, u_1, \ldots, u_\ell \in X^*$, $a_1, a_2, \ldots, a_\ell \in X$ and $u_0 a_1 u_1 \ldots a_\ell u_\ell = u$.

The set of all factorizations of lengths at most k of the word u is denoted by $\mathrm{Fact}_k(u)$.

Main construction

For a factorization $f = (u_0, a_1, u_1, \ldots, a_\ell, u_\ell)$ of a word $u \in X^*$ and a factorization $g = (v_0, b_1, v_1, \ldots, b_m, v_m)$ of a word $v \in X^*$, we write $\quad f \leq_\gamma g \quad$ if

- $\ell = m$,
- $a_i = b_i$ for every $i \in \{1, \ldots, \ell\}$,
- $u_i \; \gamma \; v_i$ for every $i \in \{0, 1, \ldots, \ell\}$.

We define the relation $\mathsf{p}_k(\gamma)$ on the set X^* as follows:
for $u, v \in X^*$, we have $\quad (u, v) \in \mathsf{p}_k(\gamma) \quad$ iff

$$(\forall\, g \in \mathrm{Fact}_k(v))\; (\exists\, f \in \mathrm{Fact}_k(u))\; f \leq_\gamma g \,.$$

Main theorem - CAI

Theorem

Let $\mathscr{V}$ *be a locally finite positive variety of languages and* γ *be a finite characteristic of* $\mathscr{V}$. *Then* $\mathsf{PPol}_k\mathscr{V}$ *is a locally finite positive variety of languages with the finite characteristic* $\mathsf{p}_k(\gamma)$ *and* $\mathsf{BPol}_k\mathscr{V}$ *is a locally finite boolean variety of languages with the finite characteristic* $\mathsf{p}_k(\gamma) \cap (\mathsf{p}_k(\gamma))^{-1}$.

Basic examples

$$
\begin{array}{cccc}
 & & \cap,\ \cup & \cap,\ \cup,\ \text{compl.} \\
\mathscr{S} & \overline{B_0}\, a_1 \overline{B_1}\, a_2 \dots a_\ell \overline{B_\ell} & \mathrm{PPol}_k(\mathscr{S}) \subseteq & \mathrm{BPol}_k(\mathscr{S}) \\
 & & \cup| & \cup| \\
\mathscr{S}^+ & B_0^* a_1 B_1^* a_2 \dots a_\ell B_\ell^* & \mathrm{PPol}_k(\mathscr{S}^+) \subseteq & \mathrm{BPol}_k(\mathscr{S}^+)
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$\ell \leq k,\ a_1, \dots, a_\ell \in A,\ B_0, \dots, B_\ell \subseteq A$

Varieties of languages generated by a finite number of languages - CAI

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A positive variety $\mathscr{V}$ is generated by a finite number of languages if and only if the corresponding psedovariety $\mathbf{V}$ of ordered monoids is generated by a single ordered monoid.

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For each k, the positive variety $\mathrm{PPol}_k\mathscr{S}^+$ is generated by a finite number of languages.

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The positive variety $\mathrm{PPol}_1\mathscr{S}$ is generated by a finite number of languages.

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The positive variety $\mathrm{PPol}_2\mathscr{S}$ is not generated by a finite number of languages.

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Inclusions

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The hierarchies $\mathrm{PPol}_k(\mathscr{S}^+)$, $\mathrm{PPol}_k(\mathscr{S})$, $\mathrm{BPol}_k(\mathscr{S}^+)$ *and* $\mathrm{BPol}_k(\mathscr{S})$ *are strict.*

Proposition

For each k, the varieties $\mathrm{PPol}_k(\mathscr{S}^+)$, $\mathrm{PPol}_k(\mathscr{S})$, $\mathrm{BPol}_k(\mathscr{S}^+)$ *and* $\mathrm{BPol}_k(\mathscr{S})$ *are pairwise different.*

Theorem

The only non-trivial inclusions are

$$BPol_1(\mathscr{S}) \subseteq \mathrm{PPol}_2(\mathscr{S}),\ \mathrm{BPol}_2(\mathscr{S}^+),\ \mathrm{PPol}_3(\mathscr{S}^+)$$

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Certain identities

Let x, y be two different letters from X and $u \in X^*$ be a word which contains both x and y, i.e. $x, y \in \mathrm{c}(u)$. The "identity"

$$uxyx = uyx \ , \quad \text{where} \quad x, y \in \mathrm{c}(u) \tag{1}$$

is equivalent to a pair of identities: we distinguish two cases $u = u_1 x u_2 y u_3$ and $u = u_1 y u_2 x u_3$ for some $u_1, u_2, u_3 \in X^*$, so the identity (1) is equivalent to the identities

$$x_1\, x\, x_2\, y\, x_3 \cdot x\, y\, x = x_1\, x\, x_2\, y\, x_3 \cdot y\, x \ ,$$

$$x_1\, y\, x_2\, x\, x_3 \cdot x\, y\, x = x_1\, y\, x_2\, x\, x_3 \cdot y\, x \ .$$

We have also the dual version of the identity (1)

$$xyxu = xyu \quad \text{where} \quad x, y \in \mathrm{c}(u) \ .$$

Proposition 1

Consider also

$$uxyv = uyxv \ , \quad \text{where} \quad x, y \in \mathrm{c}(u) \cap \mathrm{c}(v) \ . \tag{2}$$

Note that this identity represents in fact four identities.

$$yuyx \leq yuxyx \quad \text{and} \quad xyuy \leq xyxuy \tag{3}$$

$$xuxvx \leq xuvx \ . \tag{4}$$

Proposition

(i) The identities (1) and (2) form a finite basis of identities for the variety of monoids corresponding to $\mathrm{BPol}_1(\mathscr{S}^+)$.
(ii) The identities (2), (3) and (4) form a finite basis of identities for the variety of ordered monoids corresponding to $\mathrm{PPol}_1(\mathscr{S}^+)$.

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(i) The variety of monoids corresponding to $\mathrm{BPol}_1(\mathscr{S})$ *has a finite basis of identities.*
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