# ON PARALLELOGRAM LAWS FOR SKEW LATTICES

**João Pita Costa**
joaopitacosta@gmail.com

in a joint work with Karin Cvetko-Vah

Department of Mathematics
University of Ljubljana

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# Defining Skew Lattices

A skew lattice **S** is a set $S$ equipped with two associative, idempotent binary operations $\vee$ and $\wedge$ that satisfy the absorption laws $(b \wedge a) \vee a = a = a \vee (a \wedge b)$ and their duals.

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$$x \vee y = x \text{ iff } x \wedge y = y \text{ and } x \vee y = y \text{ iff } x \wedge y = x.$$

## Skew Lattices in Rings

Let **R** be a ring and $E(R)$ the set of idempotent elements in $R$. Set $x \wedge y = xy$ and $x \vee y = x + y - xy$.

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Let **R** be a ring and $E(R)$ the set of idempotent elements in $R$. Set $x \wedge y = xy$ and $x \vee y = x + y - xy$. Any normal skew lattice with a finite lattice image can be embedded into some ring of matrices [Cvetko-Vah 2005].

# Sketching Skew Lattices

In a skew lattice **S** we are able to define a natural partial ordering by $x \geq y$ iff $x \wedge y = y = y \wedge x$, or dually, $x \vee y = x = y \vee x$.

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$$xDy \text{ iff } x \vee y \vee x = x \text{ and } y \vee x \vee y = y$$

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$S/\mathcal{D}$ is the maximal lattice image of $S$ with each $\mathcal{D}$ equivalence beeing a maximal noncommutative subalgebra in **S**.

# Cancellativity and Symmetry

## Definition

A skew lattice **S** is **cancellative** if , for all $x, y, z \in S$, $x \vee y = x \vee z$ and $x \wedge y = x \wedge z$ implies $y = z$, as well as $x \vee y = z \vee y$ and $x \wedge y = z \wedge y$ implies $x = z$.

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A skew lattice **S** is **distributive** if, for all $x, y, z \in S$, $x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$ and its dual.

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**SkCan** and **SkDist** are independent.

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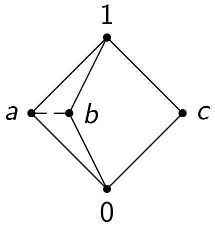
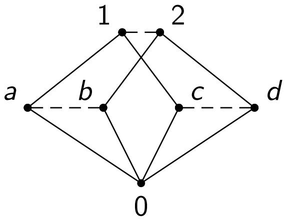
$$\mathbf{SkCan} \subsetneq \mathbf{SkSym}$$

# Forbidden Structures




$NC_5^R$




$NS_7^{R,0}$

# Coset Structure

Consider a skew lattice **S** consisting of exactly two $\mathcal{D}$-classes $\mathbf{A} > \mathbf{B}$. For some $b \in B$, a subset $A \wedge b \wedge A = \{ a \wedge b \wedge a : a \in A \}$ is said to be a coset of **A** in **B**. Similarly, a coset of **B** in **A** is any subset $B \vee a \vee B = \{ b \vee a \vee b : b \in B \}$ of $A$, for a fixed $a \in A$.

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$$a \wedge B \wedge a = \{ a \wedge b \wedge a : b \in B \} = \{ b \in B : b \leq a \}$$

is the image of $a$ in $\mathbf{B}$. Dually, for any $b \in B$, the set $b \vee A \vee b = \{ a \in A : b \leq a \}$ is its image in $\mathbf{A}$.

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(c) Given cosets $B \vee a \vee B$ in $A$ and $A \wedge b \wedge A$ in $B$, a natural bijection of cosets is given by the natural partial ordering: $x \in B \vee a \vee B$ corresponds to $y \in A \wedge b \wedge A$ if, and only if, $x \geq y$.

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(d) The meet and join on **S** are determined jointly by the coset bijections and rectangular structure of each $D$-class.

# Coset Structure

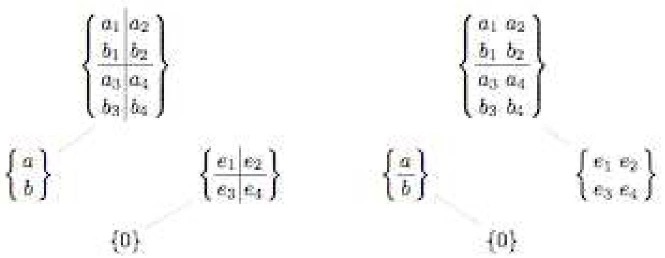


Figure: Coset Partitions on the Skew Diamond

# Coset Constants

Let **S** is a right handed skew lattice in a ring $R$. For any pair of cosets $B_j \subset B$ and $A_i \subset A$, a constant $c(B_j, A_i) \in R$ exists such that $\phi_{ji}(a) = a + c(B_j, A_i)$ for all $a \in A_i$.

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$$c(xB, M \circ y) = c(xJ, A \circ y).$$

# Parallelogram Laws for Skew Lattices in Rings

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**Theorem (Karin Cvetko-Vah 2005)**

*Let **S** is a right handed skew lattice in a ring $R$. The following coset identities in **S** hold and are equivalent:*

(i) *for all skew diamond* $\{ J > A, B > M \}$ *in* $S$ *and* $x, x' \in A$, $M \circ x = M \circ x'$ *if, and only if,* $B \circ x = B \circ x'$

(ii) *for all skew diamond* $\{ J > A, B > M \}$ *in* $S$ *and* $x, x' \in A$, $xB = x'B$ *if, and only if,* $xJ = x'J$

# Parallelogram Laws for Cancellative Skew Lattices

**Theorem (K. Cvetko-Vah & JPC 2009)**

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$M \vee x \vee M = M \vee x' \vee M$ *iff* $B \vee x \vee B = B \vee x' \vee B$

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(iii) *for all* $\{\, J > A, B > M \,\}$ *in* $S$ *and* $x, x' \in A$,
$B \wedge x \wedge B = B \wedge x' \wedge B$ *iff* $J \wedge x \wedge J = J \wedge x' \wedge J$

# Parallelogram Laws for Symmetric Skew Lattices

As for symmetric skew lattices, J. Leech characterized them as the ones satisfying the identities bellow,

(i) $J \wedge m \wedge J = (A \wedge m \wedge A) \bigcap (B \wedge m \wedge B)$, for all $m \in M$;

(ii) $M \vee j \vee M = (A \vee j \vee A) \bigcap (B \vee j \vee B)$, for all $j \in J$.

# Parallelogram Laws for Symmetric Skew Lattices

**Theorem (J. Leech 1993)**

*Let* **S** *be a skew lattice.* **S** *is symmetric if, and only if, for all skew diamonds* $\{J > A, B > M\}$ *in* $S$*, the following are satisfied:*

(i) $J \wedge m \wedge J = J \wedge m' \wedge J$ *if, and only if,* $A \wedge m \wedge A = A \wedge m' \wedge A$ *and* $B \wedge m \wedge = B \wedge m' \wedge B$;

(ii) $M \vee j \vee M = M \vee j' \vee M$ *if, and only if,* $A \vee j \vee A = A \vee j' \vee A$ *and* $B \vee j \vee B = B \vee j' \vee B$.

# Index of a Coset

When we consider a skew lattice **S** with comparable $\mathcal{D}$ classes **A** $>$ **B** we may look at the size of the existing cosets.

$$b \vee A \vee b = \{\, a \in A : a \geq b \,\}\,,\ \text{with } b \in B$$

and dually

$$a \wedge B \wedge a = \{\, b \in M : a \geq b \,\}\,,\ \text{with } a \in A$$

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All image sets of elements from one class have equal size, ie,

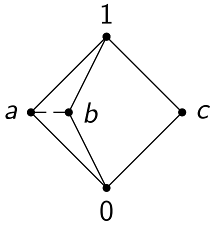
$$\text{for all } b, b' \in B,\ \left|\{\, a \in A : a \geq b \,\}\right| = \left|\{\, a \in A : a \geq b' \,\}\right|$$

# Index of a Coset

**Definition**

The index of $B$ in $A$, $[B : A] = |\{ a \in A : a \geq b \}|$, equals the number of $B$-cosets in $A$. Dually, we define the index of $A$ in $B$, $[A : B] = |\{ b \in B : a \geq b \}|$.

# Index of a Coset




$$[M:A]=2, [B:M]=1, [J:A]=2, [B:J]=1,$$
$$[A:M]=1, [M:B]=1, [A:J]=1, [J:B]=1.$$

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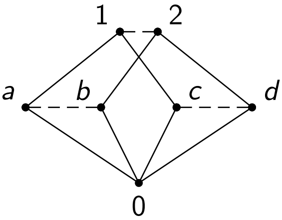


$$[M:A]=2,[B:M]=1,[J:A]=2,[B:J]=2,$$
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# Cancellative Case

## Proposition

*Let* **S** *be a cancellative skew lattice. Given* $\{\, J > A, B > M \,\}$ *in* **S**,

$[M : A] = [B : J], [B : M] = [J : A]$
*and, dually,* $[A : M] = [J : B], [M : B] = [A : J]$

# Symmetric Case

**Proposition**

*Let* **S** *be a symmetric skew lattice. Given a skew diamond* $\{ J > A, B > M \}$ *in* **S**,

$$[J : M] = [A : M][B : M] \text{ and, dually, } [M : J] = [A : J][B : J]$$

# Combinatoric Results

Given a skew lattice $\mathbf{S}$ with comparable $\mathcal{D}$ classes $\mathbf{A} > \mathbf{B}$, consider the family of images of $B$-cosets in $A$, $(A_i)_{i\in[B:A]}$ and the family of images of $A$-cosets in $B$, $(B_j)_{j\in[A:B]}$.

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$$[A:B] = \frac{[B:A].|B|}{|A|}.$$

# Symmetric Case

## Proposition

*Given a skew diamond* $\{ J > A, B > M \}$ *in a symmetric skew lattice* $\mathbf{S}$,

$$\frac{|A|.|B|}{|J|.|M|} = \frac{[M : A].[M : B]}{[M : J]}$$

# Cancellative Case

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# Thank you