Ternary differential modes

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(based on join work with A. Romanowska
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Modes

- idempotent: each singleton is a subalgebra
- entropic: all term operations commute each other

Embeddability of algebras into the others

Example

- (A.Romanowska and J.D.H.Smith) Each cancellative mode embeds as a subreduct into a module.

A mode (A, F) is cancellative if it satisfies the quasi-identity:

$$f(a_1, \dots, x_i, \dots, a_n) = f(a_1, \dots, y_i, \dots, a_n) \longrightarrow x_i = y_i.$$

for each n-ary operation $f \in F$ and each $i = 1, \cdots, n$.

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- (K.Kearnes) Each semilattice mode (a mode with a semilattice term operation) is a subreduct of a semimodule over a commutative semiring.

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for each n-ary operation $f \in F$ and each $i = 1, \cdots, n$.

Szendrei modes

Theorem (M.Stronkowski)

A mode (A, F) embeds into a semimodule over a commutative semiring with unity iff it is so-called Szendrei mode - mode satisfying Szendrei identities:

$$f(f(x_{11}, \dots, x_{1n}), \dots, f(x_{n1}, \dots, x_{nn})) =$$
$$f(f(x_{\pi(11)}, \dots, x_{\pi(1n)}), \dots, f(x_{\pi(n1)}, \dots, x_{\pi(nn)})),$$

for each n-ary operation $f \in F$ and every transposition $\pi : ij \mapsto ji$ of indices.

Ternary Szendrei modes

Example

Szendrei identities in the case of one ternary operation $f(x, y, z)$:

$$f(f(x_{11}, x_{12}, x_{13}), f(x_{21}, x_{22}, x_{23}), f(x_{31}, x_{32}, x_{33})) =$$
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Non Szendrei modes

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(D. Stanovský) The 3-elements algebra $(D = \{0, 1, 2\}, f)$ with one ternary operation $f : D^3 \to D;\ \ (x, y, z) \mapsto f(x, y, z)$

$$f(x,y,z) := \begin{cases} 2 - x, & \text{if } y = z = 1 \\ x & \text{otherwise.} \end{cases}$$

is a mode, but not Szenderi:
$((210)(000)(100)) = (201) = 2 \neq 0 = (200) = ((211)(000)(000))$.

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$(011) = 2$ and $(211) = 0$

Varieties $\mathcal{D}_3$ and $\mathcal{D}_2$

The algebra (D, f) belongs to the variety $\mathcal{D}_3$ of ternary differential modes with one ternary operation $f(x, y, z) := (xyz)$ defined by two additional identities:

$$\begin{aligned} f(f(x, y_1, y_2), z_1, z_2) &= f(f(x, z_1, z_2), y_1, y_2), \\ f(x, y_1, y_2) &= f(x, f(y_1, z_1, z_2), f(y_2, z_1, z_2)). \end{aligned}$$

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The variety $\mathcal{D}_2$ of differential groupoids is the variety of groupoid modes defined by:

$$\begin{aligned} (xy)z &= (xz)y, \\ x(yz) &= xy. \end{aligned}$$

Free ternary differential modes

Let X be a set of variables and $F_{\mathcal{D}_3}(X)$ be the free $\mathcal{D}_3$-algebra over X. For $x_i, x_j \in X$, denote the right translation $R_{x_i x_j} : F_{\mathcal{D}_3}(X) \to F_{\mathcal{D}_3}(X)$, $x \mapsto x R_{x_i x_j} := (x x_i x_j)$ by R_{ij}.

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Theorem

Each element $w(x_1, \ldots, x_n)$ of the free $\mathcal{D}_3$-algebra $F_{\mathcal{D}_3}(X)$ over a set X may be expressed in the standard form

$$x_1 R_{12}^{k_{12}} \ldots R_{1n}^{k_{1n}} R_{21}^{k_{21}} \ldots R_{2n}^{k_{2n}} \ldots R_{n1}^{k_{n1}} \ldots R_{nn}^{k_{nn}},$$

where x_1 is its leftmost variable and the indices $ij \in \{1, \ldots, n\} \times \{1, \ldots, n\}$ are ordered lexicographically.

The lattice of subvarieties of $\mathcal{D}_2$

Theorem (A.Romanowska and B.Roszkowska)

Each proper non-trivial subvariety of $\mathcal{D}_2$ is relatively based by identity of the form

$$xy^{i+k} = xy^i,$$

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The lattice of non-trivial subvarieties of $\mathcal{D}_2$ is dually isomorphic to the congruence lattice of the monoid $(\mathbb{N}, +, 0)$ of natural numbers.

Theorem

Every proper subvariety of the variety $\mathcal{D}_3$ either has an equational basis consisting of the axioms of $\mathcal{D}_3$ and one additional identity, or is non-finitely based.

Subvarieties of the variety $\mathcal{D}_3$

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Example

Let $\mathcal{V}$ be the subvariety of the variety $\mathcal{D}_3$ defined by the identity

$$xR_{yz}^3 = xR_{yz}^2$$

and for natural numbers n all the identities

$$xR_{01}R_{12}R_{23}\dots R_{n-1n}R_{nn+1} = xR_{01}R_{12}R_{23}\dots R_{n-1n}R_{n+1n}$$

(where R_{ij} is $R_{x_ix_j}$). The variety $\mathcal{V}$ is locally finite, but has no finite basis for its identities.

Free Szendrei ternary differential modes

Theorem

Each element $w(x_1, \dots, x_n)$ of the free Szendrei ternary differential mode over a set X, where the set $\{x_1, \dots, x_n\}$ may be expressed in the following standard form

$$x_i R_{i1}^{l_{i1}} R_{1i}^{l_{1i}} \dots R_{in}^{l_{in}} R_{ni}^{l_{ni}}$$

for some $l_{ij}, l_{ji} \in \mathbb{N}$ with $l_{ii} = 0$ and x_i - the leftmost variable.

The lattice of subvarieties of the variety $Sz(\mathcal{D}_3)$

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Corollary

Each variety of Szendrei differential modes has a finite basis for its identities.

Let $\mathcal{V}_{m,n}^{m',n'}$ be the subvariety of $Sz(\mathcal{D}_3)$ defined by the identity

$$xR_{xy}^{m}R_{yx}^{n} = xR_{xy}^{m'}R_{yx}^{n'}.$$

Let (i,j) ba a pair of relatively prime natural number $(i,j \neq 0)$ and $(m,n) \in \mathbb{N} \times \mathbb{N}$, $l \in \mathbb{Z}^{+}$.

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Proposition

- The varieties $\mathcal{V}_{m,n}^{m+li,n+lj}$ form a lattice isomorphic to the lattice $\mathbb{N} \times \mathbb{N} \times \mathbb{Z}^{+}$, where two first factors are with the usual linear ordering and the third is ordered by the divisibility relation.

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- The varieties $\mathcal{V}_{m+i,n}^{m,n+j}$ form a lattice isomorphic to the lattice $\mathbb{N} \times \mathbb{N}$ with the usual ordering relation.

The lattice of subvarieties of the variety $Sz(\mathcal{D}_3)$

The variety $Sz(\mathcal{D}_3)$ has a unique atom - the variety $\mathcal{LZ}_3$ of left-zero algebras relatively based by two identities:

$$(xxy) = x, \quad (xyx) = x.$$

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Theorem

Let p be a prime and $a = 0, 1, \ldots, p-1$. The following five varieties cover the variety $\mathcal{LZ}_3$:

$$\mathcal{V}^{p,0}_{0,0} \cap \mathcal{V}^{a,0}_{0,0}$$

$$\mathcal{V}^{0,p}_{0,0} \cap \mathcal{V}^{1,0}_{0,0}$$

$$\mathcal{V}^{2,0}_{1,0} \cap \mathcal{V}^{0,1}_{0,0}$$

$$\mathcal{V}^{0,2}_{0,1} \cap \mathcal{V}^{1,0}_{0,0}$$

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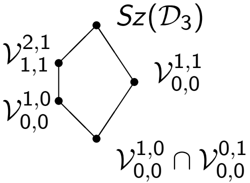
The lattice of subvarieties of $Sz(\mathcal{D}_3)$ is not modular

The lattice of varieties of Szendrei differential modes contains the following lattice isomorphic to the "pentagon" lattice N_5:



The End

Thank you for your attention