Skew order: A new look on orthomodular lattices

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Orthomodular lattices I

OMLs = “Boolean algebras which are not distributive nor uniquely complemented”:

$$x \wedge (y \vee z) \neq (x \wedge y) \vee (x \wedge z),$$

$$x \vee y = 1, x \wedge y = 0 \;\not\Rightarrow\; y = x'.$$

Orthomodular lattices II

Kröger, Beran (70's): OMLs are *skew Boolean algebras*.

$$x \mathbin{\dot\wedge} y := x \wedge (x' \vee y)$$

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OML = "BA which are not commutative nor associative":

$$x \mathbin{\dot\wedge} y \neq y \mathbin{\dot\wedge} x,$$

$$(x \mathbin{\dot\wedge} y) \mathbin{\dot\wedge} z \neq x \mathbin{\dot\wedge} (y \mathbin{\dot\wedge} z),$$

$$x \mathbin{\dot\wedge} (y \vee z) = (x \mathbin{\dot\wedge} y) \vee (x \mathbin{\dot\wedge} z),$$

$$x \mathbin{\dot\vee} y = 1, x \mathbin{\dot\wedge} y = 0 \;\Rightarrow\; y = x'.$$

Assymetry of the skew operations

Sasaki adjunction:

$$x \mathbin{\dot\wedge} y \leq z \;\Leftrightarrow\; y \leq x \mathbin{\dot\to} z \;(= x' \mathbin{\dot\vee} z),$$

$$x \mathbin{\dot\wedge} y \leq x,$$

$$x \leq y \;\Leftrightarrow\; x = y \mathbin{\dot\wedge} x,$$

$$x \mathbin{\dot\wedge} y = y \mathbin{\dot\wedge} x \;\Leftrightarrow\; xCy \text{ (then } = x \wedge y\text{).}$$

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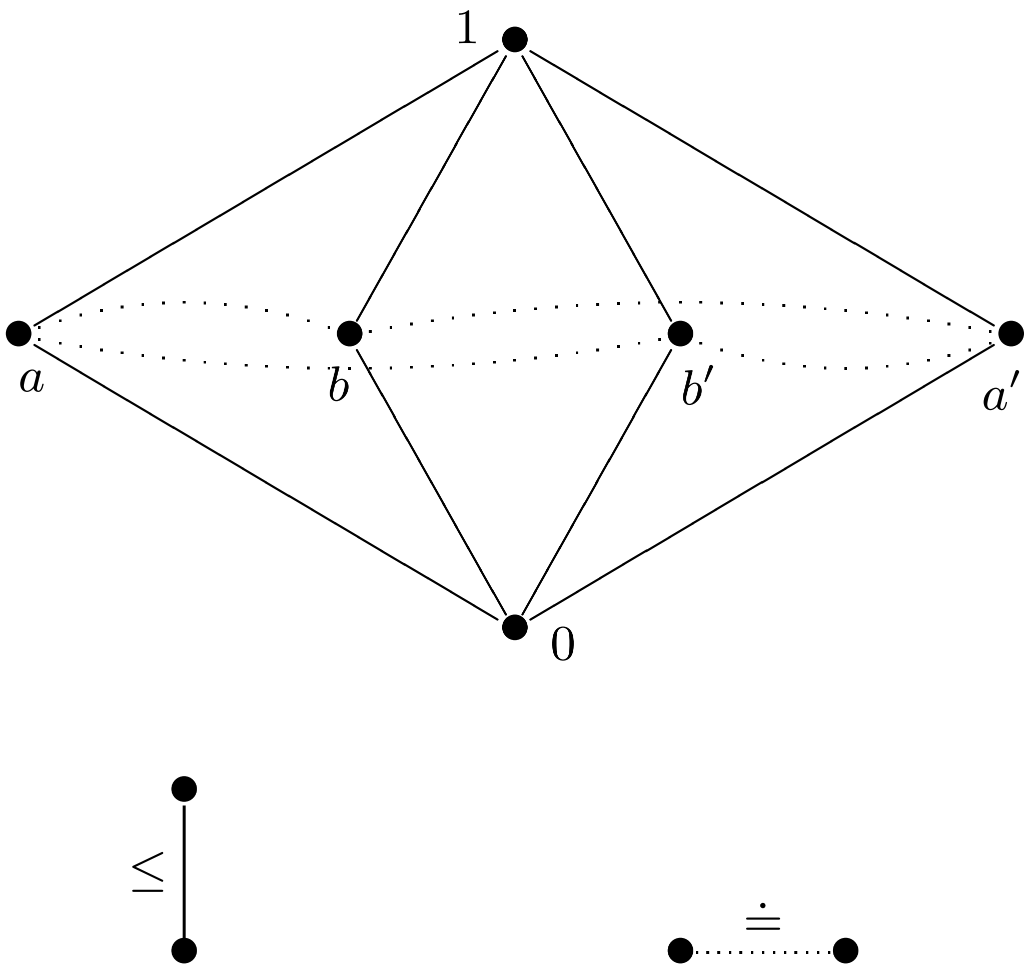
In $\mathcal{P}(H)$ (closed subspaces/projections on a Hilbert space), $x \mathrel{\dot{\leq}} y$ means that

- x is projected injectively into y,
- y is projected surjectively onto x,
- x is a "subspace" of y upto a deviation $< \pi/2$.

Properties of the skew order

- $x \mathrel{\dot{\le}} y \;\Leftrightarrow\; x \wedge y' = 0 \;\Leftrightarrow\; x' \vee y = 1 \;\Leftrightarrow\; y \mathbin{\dot{\vee}} x = y$.
- $x \mathbin{\dot{\wedge}} y \mathrel{\dot{\le}} y$.
- $\dot{\le}$ is reflexive. It is transitive or antisymmetric only in BA.
- (Skew antisymmetry) $x \le y, y \mathrel{\dot{\le}} x \;\Rightarrow\; x = y$.
- (Skew transitivity) $x \le y \mathrel{\dot{\le}} z$ or $x \mathrel{\dot{\le}} y \le z \;\Rightarrow\; x \mathrel{\dot{\le}} z$.
- $x \le y \;\Leftrightarrow\; x \mathrel{\dot{\le}} y, xCy$. In particular, $\dot{\le}$ contains $\le$.
- When $x \doteq y \;:\Leftrightarrow\; x \mathrel{\dot{\le}} y, y \mathrel{\dot{\le}} x$, then $(\dot{\le}) = (\doteq) \circ (\le) = (\le) \circ (\doteq)$. Namely, for $x \mathrel{\dot{\le}} y$ we have $x \doteq x \mathbin{\dot{\wedge}} y \le y$ and $x \le y \mathbin{\dot{\vee}} x \doteq y$.

Example



Uniqueness

Theorem. Let $x\rho y$ be given by identities in language $(\wedge, \vee, 0, 1, ')$ and variables x, y such that ρ is $\leq$ on a Boolean algebra. Then, in any OML, ρ is either $\leq$ or $\dot{\leq}$.

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Proof.

$$\begin{tikzcd}
& \{l_i = r_i\} \arrow[r, "(=)=(\leq)\cap(\geq)"] & \{l_j \leq r_j\} \arrow[r, "\text{Sasaki adjunction}"] & \{l_k \leq 0\} \arrow[d, "\bigvee_k"] \\
l = x \mathbin{\dot\wedge} y' \text{ or } l = x \wedge y' & & & l = 0 \arrow[lll, "\text{Pavičić, Megill}"']
\end{tikzcd}$$

Skew adjointness

Theorem. Considering all 6 quantum meets and all 6 quantum implications, the Sasaki adjunction

$$x \mathbin{\dot\wedge} y \leq z \;\Leftrightarrow\; y \leq x \mathbin{\dot\rightarrow} z$$

and the *skew adjunction*

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(where $x \rightarrow z = x' \vee z$) are the only ones valid for all OML.

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Proof. An easy manipulation with 6-element OML $\{0, a, a', b, b', 1\}$ eliminates other possibilities. The skew adjunction is obvious, since both sides are equivalent to $x \wedge y \wedge z' = 0$.

Skew nuclei

$$j(x \wedge y) = j(x) \wedge j(y),$$

$$j^2 = j,$$

$$x \mathrel{\dot{\leq}} j(x).$$

Theorem. By ordering nuclei $j \leq k :\Leftrightarrow\ (\forall x)(j(x) \mathrel{\dot{\leq}} k(x))$ we get an isomorphism $L \cong N(L)$ between OML L and lattice $N(L)$ of all skew nuclei.

Proof. Every skew nucleus is of the form $c_a(x) = a \mathbin{\dot{\vee}} x$ for some $a \in L$.

Applications and problems

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Given a (nc.) C*-algebra A, $(\text{RIdl}\,(A), \leq, \dot\leq)$ is a full invariant of A. TFAE:
(i) RIdl (A) is closed under $\dot\wedge$,
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Skew orders in non-orthomodular lattices?