The finite congruence lattice problem
3. Some constructions

Péter P. Pálfy
Alfréd Rényi Institute of Mathematics,
Hungarian Academy of Sciences
and
Eötvös University, Budapest

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Partition lattices

Obviously, the **partition lattice** $\mathrm{Part}(k)$ (the lattice of all equivalence relations on a k-element set) is a congruence lattice.

It is also an interval in a subgroup lattice, for example

$$\mathrm{Int}(S_1 \times S_2 \times S_4 \times \cdots \times S_{2^{k-1}}; S_{2^k-1}) \cong \mathrm{Part}(k).$$

Lemma. Let S be a finite nonabelian simple group, and $D = \{(s, s, \ldots, s) | s \in S\}$ the diagonal subgroup in S^k.
Then $\mathrm{Int}(D; S^k)$ is the dual of $\mathrm{Part}(k)$.

Proof. Every subgroup $D \leq X \leq S^k$ is a subdirect power of the form $\{(s_{i(1)}^{\alpha_1}, s_{i(2)}^{\alpha_2}, \ldots, s_{i(k)}^{\alpha_k}) | s_1, \ldots, s_m \in S\}$, where $i : \{1, \ldots, k\} \to \{1, \ldots, m\}$ and $\alpha_1, \ldots, \alpha_k \in \mathrm{Aut}(S)$. Since $D \leq X$, all automorphisms can be taken to the identity.
For example $X = \{(s_1, s_2, s_1, s_1, s_3, s_2) | s_1, s_2, s_3 \in S\}$.
So X is determined by the kernel of the mapping i.
The larger the kernel of i is, the smaller is the corresponding subgroup.

The dual lattice

Theorem (Kurzweil, 1985; Netter)
The dual of a finitely representable lattice is also finitely representable.
Proof. Let $L \cong \mathrm{Con}(U; F)$ for a unary algebra $(U; F)$. Take any finite nonabelian simple group S.
Take the permutation group (unary algebra) $(S^U : D; S^U)$, its congruence lattice is the dual of $\mathrm{Part}(U)$.
The elements of S^U are functions $U \to S$, the elements of the diagonal subgroup D are the constant functions.
The operations $f \in F$, $f : U \to U$ give rise to operations on S^U simply by composition: if $g : U \to S$, then $f(g) : U \to S$ is defined by $(f(g))(u) = g(f(u))$.
If we multiply g by a constant, then $f(g)$ will be multiplied by the same constant, therefore f can be defined on $S^U : D$ as well.
A congruence of $(S^U : D; S^U)$ remains a congruence of the algebra $(S^U : D; S^U \cup F)$ iff it corresponds to a partition invariant under all $f \in F$, that is, iff it is a congruence of $(U; F)$.

Intervals and sublattices

If $\vartheta \in \mathrm{Con}(U; F)$, then $\mathrm{Con}(U/\vartheta; F) \cong \mathrm{Int}(\vartheta; 1)$, so a filter in the congruence lattice is again a congruence lattice.

The theorem about the representation of the dual lattice then yields:
Corollary. Every interval is a finitely representable lattice is also finitely representable.
John Snow (2000) gave a direct proof.

Is every sublattice of a finitely representable lattice also finitely representable?
Theorem (Pudlák and Tůma, 1980)
Every finite lattice can be embedded into a suitable finite partition lattice.

HSP

Is every homomorphic image of a finitely representable lattice also finitely representable?

Lemma (P[5], 1980) Let $e \in \mathrm{Pol}_1(U; F)$ be an idempotent function ($e^2 = e$), then the restriction is a lattice homomorphism of $\mathrm{Con}(U; F)$ $\underline{\text{onto}}$ $\mathrm{Con}(e(U); eF)$ (the **induced algebra**).

Lemma. The direct product of finitely representable lattices is also finitely representable.
Proof. Take the product of transformation monoids containing all constants, then
$\mathrm{Con}(U_1 \times U_2; F_1 \times F_2) = \mathrm{Con}(U_1; F_1) \times \mathrm{Con}(U_2; F_2)$.
Here $(f_1, f_2)(u_1, u_2) = (f_1(u_1), f_2(u_2))$.

Snowmobile-1

Lemma 1 (Snow, 2000) Let $\alpha, \beta \in \mathrm{Con}(U; F)$. Then we can find additional operations F^* so that

$$\mathrm{Con}(U; F \cup F^*) = \{\gamma \in \mathrm{Con}(U; F) | \gamma \leq \alpha \,\mathrm{or}\, \gamma \geq \beta\}.$$

Proof. Let F^* consist of those unary operations whose kernel contains α and the image lies in one β-class.
If $\alpha \geq \gamma \in \mathrm{Con}(U; F)$, $f^* \in F^*$ and $(u, v) \in \gamma \leq \alpha$, then $f^*(u) = f^*(v)$, so f^* preserves γ.
If $\beta \leq \gamma \in \mathrm{Con}(U; F)$, $f^* \in F^*$ (and $(u, v) \in \gamma$), then $f^*(u)$ and $f^*(v)$ lie in the same β-class, so in the same γ-class, hence f^* preserves γ.
If $\gamma \in \mathrm{Con}(U; F)$ is such that $\alpha \not\geq \gamma$ and $\beta \not\leq \gamma$, then choose $(u, v) \in \gamma \setminus \alpha$ and $(u', v') \in \beta \setminus \gamma$. Let f^* take the value u' on the α-class of u and v' everywhere else. Then $f^* \in F^*$, $(u, v) \in \gamma$, but $(f^*(u), f^*(v)) = (u', v') \notin \gamma$.

Snowmobile-2

Lemma 2 (Snow) Let $\beta_1 \leq \alpha_1$, $\beta_2 \leq \alpha_2$ be congruences of $(U; F)$ such that $\beta_1 \vee \beta_2 = 1$ and $\alpha_1 \wedge \alpha_2 = 0$. Then we can find additional operations F^* so that

$$\mathrm{Con}(U; F \cup F^*) = \{0\} \cup \mathrm{Int}(\beta_1; \alpha_1) \cup \mathrm{Int}(\beta_2; \alpha_2) \cup \{1\}.$$

Proof. Take the additional operations provided by Lemma 1 both for the pair β_1, α_2 and for β_2, α_1. Then the congruences that remain are those which lie

(above β_1 or below α_2) and (above β_2 or below α_1),

that is

$$\gamma \geq \beta_1 \vee \beta_2 \text{ or } \beta_1 \leq \gamma \leq \alpha_1 \text{ or } \beta_2 \leq \gamma \leq \alpha_2 \text{ or } \gamma \leq \alpha_1 \wedge \alpha_2.$$

More Snow (1)

Theorem (Snow, 2000) The ordinal sum and the parallel sum of two finitely representable lattices are also finitely representable.

Proof. The **ordinal sum** of L_1 and L_2 is their disjoint union, where every element of L_1 is smaller than each element of L_2. A somewhat more natural version of the ordinal sum of two lattices is obtained from the usual ordinal sum if we identify the largest element of L_1 with the smallest element of L_2.
This construct will be denoted by $L_1 + L_2$.
(A noncommutative—but associative—addition!)
The usual ordinal sum of L_1 and L_2 is just $L_1 + \mathbf{2} + L_2$.
Now take a finite algebra with conguence lattice $L_1 \times L_2$ and use Snowmobile-1 with $\alpha = \beta = (1, 0)$. Then we obtain a finite algebra with congruence lattice $L_1 + L_2$.

More Snow (2)

The **parallel sum** of L_1 and L_2 is the disjoint union

$$\{0\} \cup L_1 \cup L_2 \cup \{1\},$$

where the elements of L_1 and L_2 are pairwise incomparable.
First we prove the claim when L_2 is the 1-element lattice, and we will denote the parallel sum of L and the 1-element lattice by L^+. (It has three additional elements: $0 < m < 1$.)
Let $\mathrm{Con}(U; F) \cong L$. Take the algebra $(U \times \{1, 2\}; F)$, where $f(u, i) = (f(u), i)$. Use Snowmobile-2 with the following congruences: α_1 has two classes $U \times \{1\}$ and $U \times \{2\}$, β_1 has one nonsingleton class $U \times \{1\}$, $\alpha_2 = \beta_2$ has 2-element classes $\{(u, 1), (u, 2)\}$.
So we obtain an algebra with congruence lattice isomorphic to L^+.
In general, the parallel sum of L_1 and L_2 can be obtained using Snowmobile-2 in the congruence lattice $L_1^+ \times L_2^+$ with $\alpha_1 = (1_1, m)$, $\beta_1 = (0_1, m)$, $\alpha_2 = (m, 1_2)$, $\beta_2 = (m, 0_2)$.

Some classes of finitely representable lattices

Definition A finite(ly generated) lattice L is **lower bounded** if there exists an epimorphism $\varphi : FL(X) \to L$ such that $\forall a \in L : \{w \in FL(X) | \varphi(w) \geq a\}$ has a least element.

Theorem. A finite lattice L is lower bounded iff L and $\mathrm{Con}(L)$ has the same number of join irreducible elements.

Theorem (Pudlák and Tůma, 1976)
The finite lower bounded lattices are finitely representable.
(They called these lattices **finitely fermentable**.)

Theorem (Snow, 2000) Every finite lattice which contains no three element antichains is finitely representable.

Theorem (Snow, 2003) Every finite lattice in the variety generated by M_3 is finitely representable.

Hereditary congruence lattices

The idea of Snow's proof is this:
If L is a finite lattice in the variety generated by M_3, then L is a 0–1-sublattice of M_3^k for some k. $M_3 \cong \mathrm{Part}(3)$, so L can be considered as a 0–1-sublattice of $\mathrm{Part}(3)^k \subset \mathrm{Part}(3^k)$. He then proves that every 0–1-sublattice of $\mathrm{Part}(3)^k$ is the congruence lattice of some algebra on the 3^k-element set.

Definition (Hegedűs and P^3, 2005) A 0–1-sublattice L of all equivalence relations on a finite set U is called a **hereditary congruence lattice** if every 0–1-sublattice $L' \subseteq L$ is the congruence lattice of a suitable algebra on U. Furthermore, L is called **power-hereditary** if L^k as a lattice of equivalence relations on U^k is a hereditary congruence lattice for every $k \geq 1$.

In this language Snow's result says that the lattice of all equivalences on the 3-element set is power-hereditary.

Snakes

$\mathrm{Con}(Z_2 \times Z_2)$ ($\cong M_3$) is also power-hereditary (Hegedűs and P^3), but there are non-power-hereditary representations of M_3 as well (P^3, 2006).

Problem. Is there a hereditary congruence lattice isomorphic to M_4? That is $\mathrm{Con}(U; F) \cong M_4$ and for every nontrivial congruence ϑ_i ($i = 1, \ldots, 4$) there is a unary function f_i^* such that $\mathrm{Con}(U, F \cup \{f_i^*\}) = \mathrm{Con}(U; F) \setminus \{\vartheta_i\}$.

A **snake** of length $n \geq 2$ is a modular lattice glued together from $n - 1$ M_3's.

Theorem (Hegedűs and P^3, 2005) Every finite lattice in the variety generated by all snakes is finitely representable.

We construct operator groups $(A; +, F)$, where $(A; +)$ is an elementary abelian 2-group and F is a suitable ring of endomorphisms of $(A; +)$.