“Basic” Algebras

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Ivan's "Basic" Algebras

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Basic algebras

Definition

A **basic algebra** is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ that satisfies the identities

$$x \oplus 0 = 0,$$
$$\neg\neg x = x,$$
$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x,$$
$$\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg 0.$$

Do not confuse with Hájek's basic logic and BL-algebras!
The intersection of our basic algebras and BL-algebras are just MV-algebras.

Sectionally pseudocomplemented lattices = a non-distributive generalization of relatively pseudocomplemented lattices:

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A **sectionally pseudocomplemented lattice** is a lattice with greatest element such that every section is a pseudocomplemented lattice.

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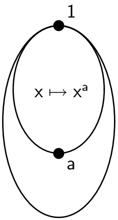


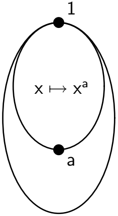
We can define the binary operations $\diamond$ and $\rightarrow$ by

$$x \diamond y := x^{x \wedge y} \quad \text{and} \quad x \rightarrow y := (x \vee y)^{y}$$

and regard sectionally pseudocomplemented lattices as algebras $(A, \vee, \wedge, \diamond, 1)$ or $(A, \vee, \wedge, \rightarrow, 1)$.
For $x \geq a$ we have

$$x^{a} = x \diamond a = x \rightarrow a.$$

If the relative pseudocomplement $x * y$ of x w.r.t. y exists, then $x * y = x \diamond y = x \rightarrow (x \wedge y)$. Indeed, $x \wedge (x \diamond y) = x \wedge y$ implies $x \diamond y \leq x * y$, and $x \wedge (x * y) = x \wedge y$ yields $x * y \leq x \diamond y$.



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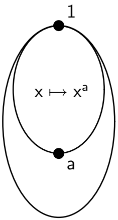

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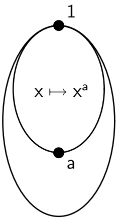

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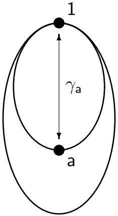
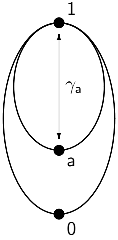


Basic algebras = bounded lattices with sectional antitone involutions, i.e., every section $[a)$ is equipped with an antitone involution γ_a.
We shall write x^a instead of $\gamma_a(x)$.



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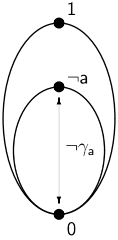

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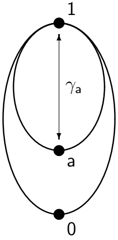

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- Orthomodular lattices: If $(A, \vee, \wedge, ', 0, 1)$ is an OML, then $\boxed{x \mapsto x' \vee a}$ is an antitone involution on $[a)$.
- MV-algebras: An MV-algebra $(A, \oplus, \neg, 0)$ is a commutative monoid $(A, \oplus, 0)$ with a unary operation $\neg$ satisfying the identities

$$\neg\neg x = x,$$
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- The equivalent algebraic semantics for the Łukasiewicz many-valued propositional logic.
- The variety of MV-algebras is generated by the standard MV-algebra $([0,1], \oplus, \neg, 0)$ where

$$x \oplus y := \min\{1, x + y\} \quad \text{and} \quad \neg x := 1 - x.$$

- All MV-algebras can be obtained as follows:
Let $(G, +, 0, \vee, \wedge)$ be an Abelian lattice-ordered group and $u \in G^+$. Then $([0,u], \oplus, \neg, 0)$ is an MV-algebra where

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Then $(A, \vee, \wedge, 0, 1)$, where $1 := \neg 0$, is a bounded lattice whose underlying order is given by

$$x \leq y \quad \text{iff} \quad \neg x \oplus y = 1,$$

and for each $a \in A$, the map

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OML's = idempotent basic algebras satisfying the quasi-identity

$$x \leq y \quad \Rightarrow \quad y \oplus x = y.$$

Proof:

- $\neg x$ is a complement of x iff $x \oplus x = x$.
  Indeed, $x \oplus x = (\neg x \vee x)^x = x$ iff $\neg x \vee x = 1$.
- If A is an OML, then $x \leq y$ implies
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  If A is not an OML, then it contains

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Basic algebras
Orthomodular lattices

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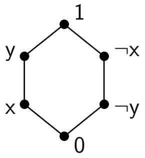
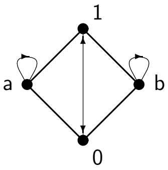


where $y \oplus x = (\neg y \vee x)^x = 1^x = x$.

The smallest basic algebra which is neither an OML nor an MV-algebra:



$\oplus$	0	a	b	1	$\neg$
0	0	a	b	1	1
a	a	1	b	1	a
b	b	a	1	1	b
1	1	1	1	1	0

$$a \oplus b = (\neg a \vee b)^b = 1^b = b$$

The term operation $\ominus$ defined by

$$x \ominus y := \neg(y \oplus \neg x)$$

is useful. We have:

- $x \oplus y = 1 \ominus ((1 \ominus y) \ominus x)$ and $\neg x = 1 \ominus x$;
- $x \leq y$ iff $x \ominus y = 0$;
- $x \vee y = (\neg y \ominus \neg x) \oplus y$ and $x \wedge y = x \ominus (x \ominus y)$;
- $(x \wedge y) \oplus z = (x \oplus z) \wedge (y \oplus z)$;
- $x \ominus (y \wedge z) = (x \ominus y) \vee (x \ominus z)$.

If the lattice is distributive, then

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- For every $a \in A$, the map

$$x \mapsto a \ominus x$$

is an antitone involution on $[0, a]$;

- $[0, a]$ is a basic algebra when equipped with the operations defined by

$$x \oplus_a y := a \ominus ((a \ominus y) \ominus x) \quad \text{and} \quad \neg_a x := a \ominus x.$$

Observe that $x \ominus_a y := \neg_a(y \oplus_a \neg_a x) = x \ominus y$ for $x, y \in [0, a]$.

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is an antitone involution on $[0, a]$;
- $[0, a]$ is a basic algebra when equipped with the operations defined by
$$x \oplus_a y := a \ominus ((a \ominus y) \ominus x) \quad \text{and} \quad \neg_a x := a \ominus x.$$
Observe that $x \ominus_a y := \neg_a(y \oplus_a \neg_a x) = x \ominus y$ for $x, y \in [0, a]$.

Theorem

The variety of basic algebras is congruence regular and arithmetical.

Regularity:

$$t_1(x, y, z) = (x \ominus y) \vee (y \ominus x) \vee z$$
$$t_2(x, y, z) = z \ominus ((x \ominus y) \vee (y \ominus x))$$

Arithmeticity:

$$m(x, y, z) = (x \ominus (y \ominus z)) \vee (z \ominus (y \ominus x)) \vee (x \wedge z)$$

Theorem

The underlying lattices of commutative basic algebras are distributive.

Proof: If A contains a copy of N_5, then

and we have $a = 1^a = (\neg c \vee a)^a = c \oplus a = a \oplus c = (\neg a \vee c)^c = (\neg u)^c = (\neg b \vee c)^c = b \oplus c = c \oplus b = (\neg c \vee b)^b = 1^b = b$.
The case when A contains M_3 is analogous.

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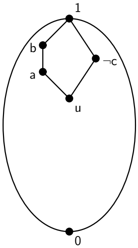

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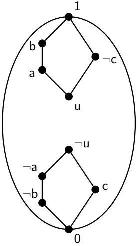

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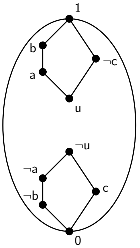

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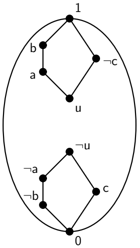

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Theorem

Every commutative basic algebra has the Riesz decomposition property:

$$x \leq a \oplus b \quad \Rightarrow \quad x = a_1 \oplus b_1 \qquad \text{(RDP)}$$

for some $a_1 \leq a$ and $b_1 \leq b$.

Proof: Put $a_1 := x \ominus b = x \ominus (x \wedge b)$ and $b_1 := x \wedge b$. Then $a_1 \leq (a \oplus b) \ominus b = a \wedge \neg b \leq a$ and

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Lemma

Every element of A is in the form

$$\bigvee_{a \in M} n_a \otimes a, \tag{F}$$

where M is the set of the atoms of A, and $n_a \in \mathbb{N}_0$ for all $a \in M$.

Here $n \otimes x := \underbrace{x \oplus \cdots \oplus x}_{n \text{ times}}$ for $n \in \mathbb{N}$, and $0 \otimes x := 0$.

Proof: Fact: If $x \wedge y = 0$, then $x \oplus y = x \vee y$, and $(m \otimes x) \wedge (n \otimes y) = 0$ for all $m, n \in \mathbb{N}_0$.

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Proof: <u>Fact</u>: If $x \wedge y = 0$, then $x \oplus y = x \vee y$, and $(m \otimes x) \wedge (n \otimes y) = 0$ for all $m, n \in \mathbb{N}_0$.

Suppose there is $z \in A$ which is not of the form (F). Then there exists $x \in A$ that is maximal among the elements which can be written in the form (F) and are less than or equal to z. Let $x = \bigvee_{a \in M} n_a \otimes a$. Further, there exists $y \in A$ such that $x \prec y \leq z$. Obviously, $b := y \ominus x$ is an atom and y is not in the form (F). Then

$$y = (y \ominus x) \oplus x = b \oplus \left(\bigvee_{a \in M} n_a \otimes a \right) = \bigvee_{a \in M} b \oplus (n_a \otimes a).$$

But for $a \neq b$ we have $b \oplus (n_a \otimes a) = b \vee (n_a \otimes a)$, so

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- For $a \in M$ the set $N(a) = \{n \otimes a \mid n \in \mathbb{N}_0\}$ is a finite chain $0 < a < \cdots < \hat{a}$.
- The RDP entails $N(a) = [0, \hat{a}]$.
- $(N(a), \oplus_{\hat{a}}, \neg_{\hat{a}}, 0)$ is a basic algebra in which $\ominus_{\hat{a}}$ coincides with the original $\ominus$ in A.
- $(N(a), \oplus_{\hat{a}}, \neg_{\hat{a}}, 0)$ is a linearly ordered MV-algebra.

Theorem

The map

$$(x_a)_{a \in M} \mapsto \bigvee_{a \in M} x_a$$

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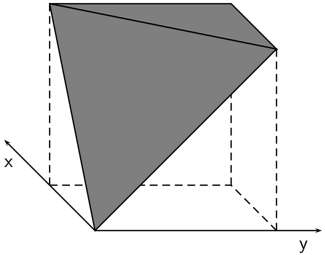
The standard MV-algebra is the algebra $([0,1], \oplus, \neg, 0)$, where

$$x \oplus y := \min\{1, x+y\} \quad \text{and} \quad \neg x := 1 - x.$$



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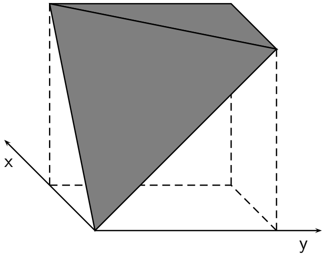

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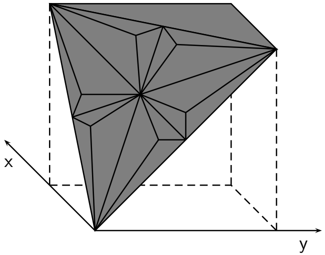

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Theorem

Every complete (as a lattice) commutative basic algebra is a subdirect product of linearly ordered commutative basic algebras.

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Two problems

1. Is every commutative basic algebra a subdirect product of linearly ordered ones?
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2. Find an associative basic algebra that is not an MV-algebra.

Definition

An **effect algebra** is a structure $(E, +, 0, 1)$ where $0, 1$ are elements of E and $+$ is a partial binary operation on E, satisfying the following conditions:

(EA1) $x + y = y + x$ if one side is defined,

(EA2) $x + (y + z) = (x + y) + z$ if one side is defined,

(EA3) for every x there exists a unique x' such that $x' + x = 1$,

(EA4) $x + 1$ is defined only for $x = 0$.

The underlying order:

$$x \leq y \quad \text{iff} \quad y = x + z \text{ for some } z;$$

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A **D-poset** is a structure $(D, \leq, -, 0, 1)$ where $(D, \leq, 0, 1)$ is a bounded poset and $-$ is a partial binary operation such that $x - y$ is defined iff $x \geq y$, satisfying the conditions

(DP1) $x - 0 = x$,

(DP2) if $x \leq y \leq z$, then $z - y \leq z - x$ and $(z - x) - (z - y) = y - x$.

To a D-poset $(D, \leq, -, 0, 1)$ there corresponds the effect algebra $(D, +, 0, 1)$ obtained by letting

$$x + y := z \quad \text{iff} \quad z \geq y \text{ and } z - y = x.$$

Lattice effect algebras/**D-lattices** are those with the underlying lattice order.

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Lattice effect algebras/**D-lattices** are those with the underlying lattice order.

In each effect algebra/D-poset:

- $x \mapsto x' + a$ is an antitone involution on $[a, 1]$,
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Hence lattice effect algebras/D-lattices are basic algebras:

Theorem

Let $(E, +, 0, 1)$ be a lattice effect algebra. If we set

$$x \oplus y := (x \wedge y') + y \quad \text{and} \quad \neg x := x',$$

then $(E, \oplus, \neg, 0)$ is a basic algebra.

Proof: $x \oplus y := (x^0 \vee y)^y = (x' \vee y)' + y = (x \wedge y') + y$.

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We have

- $x \ominus y := \neg(y \oplus \neg x) = x - (x \wedge y)$;
- $x + y = x \oplus y$ for $x \leq \neg y$;
- $x - y = x \ominus y$ for $x \geq y$.

Theorem

Let $(A, \oplus, \neg, 0)$ be a basic algebra, and define the partial operation $+$ as follows:

$$x + y \text{ is defined iff } x \leq \neg y \text{, in which case } x + y := x \oplus y.$$

Then $(A, +, 0, 1)$ is a lattice effect algebra if and only if $(A, \oplus, \neg, 0)$ satisfies the quasi-identity

$$x \leq \neg y \quad \& \quad x \oplus y \leq \neg z \quad \Rightarrow \quad (x \oplus y) \oplus z = x \oplus (z \oplus y). \quad \text{(E)}$$

We have

- $x \ominus y := \neg(y \oplus \neg x) = x - (x \wedge y)$;
- $x + y = x \oplus y$ for $x \leq \neg y$;
- $x - y = x \ominus y$ for $x \geq y$.

Theorem

Let $(A, \oplus, \neg, 0)$ be a basic algebra, and define the partial operation $+$ as follows:

$$x + y \text{ is defined iff } x \leq \neg y, \text{ in which case } x + y := x \oplus y.$$

Then $(A, +, 0, 1)$ is a lattice effect algebra if and only if $(A, \oplus, \neg, 0)$ satisfies the quasi-identity

$$x \leq \neg y \quad \& \quad x \oplus y \leq \neg z \quad \Rightarrow \quad (x \oplus y) \oplus z = x \oplus (z \oplus y). \quad \text{(E)}$$

Theorem

Let $(A, \oplus, \neg, 0)$ be a basic algebra, and define the partial operation $-$ as follows:

$$x - y \text{ is defined iff } x \geq y, \text{ in which case } x - y := x \ominus y.$$

Then $(A, \leq, -, 0, 1)$ is a D-lattice if and only if $(A, \oplus, \neg, 0)$ satisfies the quasi-identity

$$x \leq y \leq z \quad \Rightarrow \quad (z \ominus x) \ominus (z \ominus y) = y \ominus x. \qquad \text{(E')}$$

Effect basic algebras

Definition

We call a basic algebra an **effect basic algebra** if it satisfies (E) (equivalently, (E')).

Effect basic algebras (= lattice effect algebras = D-lattices) form a variety. This variety is

- congruence regular and arithmetical;
- an ideal variety; the ideal terms (in y's) are

$$t_1(x, y_1, y_2) = x \wedge (y_1 \oplus y_2),$$
$$t_2(x, y) = \neg x \ominus \neg y.$$

In a lattice effect algebra, two elements x, y are **compatible** if

$$(x \vee y) - y = x - (x \wedge y).$$

Theorem

Let $(E, \oplus, \neg, 0)$ be an effect basic algebra and $(E, +, 0, 1)$ the associated lattice effect algebra. Then $x, y \in E$ are compatible iff $x \oplus y = y \oplus x$.

Theorem

For every effect basic algebra E, the following are equivalent:

1. E is an MV-algebra;
2. E is commutative;
3. E satisfies the RDP.

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A **block** is a maximal subset whose elements commute.
The **MV-centre** is the intersection of the blocks.

Theorem

For every basic algebra E, the following are equivalent:

1. E is an effect basic algebra;
2. every block of E is a subalgebra which itself is an MV-algebra.

Theorem

Let E be an effect basic algebra. If E is subdirectly irreducible, then its MV-centre $MV(E)$ is a subdirectly irreducible MV-algebra (hence $MV(E)$ is linearly ordered).

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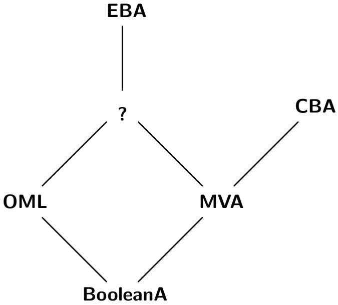
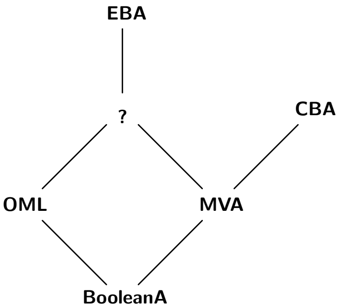
Effect basic algebras

Some varieties



Theorem

The variety generated by the algebra from Example 1 is axiomatized, relative to the variety of distributive EBA's, by the identity $(x \ominus y) \ominus (z \oplus z) = (x \ominus (z \oplus z)) \ominus (y \ominus (z \oplus z))$.



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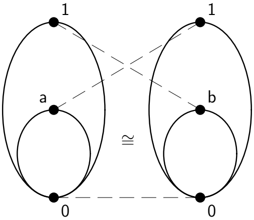
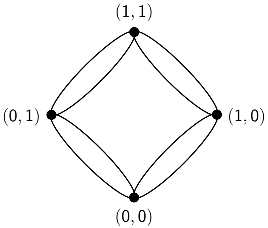
- Let A and B be σ-complete Boolean algebras. If A is isomorphic to $[0,a] \subseteq B$ and B is isomorphic to $[0,b] \subseteq A$, then $A \cong B$.
- Let A and B be σ-complete MV-algebras. If A is isomorphic to $[0,a] \subseteq B$ and B is isomorphic to $[0,b] \subseteq A$ where a, b are complemented elements, then $A \cong B$.





Definition

We say that $a \in A$ is a **central element** in a basic algebra A if

$$a = f^{-1}(0,1) \quad \text{or} \quad a = f^{-1}(1,0)$$

for some direct product decomposition $f\colon A \cong A_1 \times A_2$.
The **centre** of A, $C(A)$, is the set of all central elements.

- $C(A)$ is a subalgebra of A and a Boolean algebra in its own right.
- If A is a commutative basic algebra, then $a \in C(A)$ iff a is complemented iff $\neg a$ is a complement of a.
- If A is an effect basic algebra, then $a \in C(A)$ iff $\neg a$ is a complement of a and $a \in MV(A)$.

Cantor-Bernstein type theorem

Let A, B be basic algebras satisfying *certain conditions*. If

- $A \cong [0, a] \subseteq B$ for some $a \in C(B)$ and
- $B \cong [0, b] \subseteq A$ for some $b \in C(A)$,

then $A \cong B$.

Cantor-Bernstein theorem

Let $\mathcal{K}$ be a $\mathcal{K}$-congruence distributive quasivariety. We shall say that an algebra $A \in \mathcal{K}$ satisfies the condition $\mathscr{P}$ if for every countable set $\{\theta_i \mid i \in I\}$ of factor $\mathcal{K}$-congruences of A such that $\theta_j \circ \theta_k = \nabla_A$ for all $j \neq k$, the congruence

$$\theta_\infty := \bigcap_{i \in I} \theta_i$$

is a factor $\mathcal{K}$-congruence of A and

$$A/\theta_\infty \cong \prod_{i \in I} A/\theta_i.$$

Theorem

Let A and B be two algebras in $\mathcal{K}$ satisfying the condition $\mathscr{P}$. If

$$A \cong B \times C \quad \text{and} \quad B \cong A \times D$$

for some $C, D \in \mathcal{K}$, then $A \cong B$.

Cantor-Bernstein theorem

Lemma

Let $A \in \mathcal{K}$ and ϕ be a factor $\mathcal{K}$-congruence of A. Then $\theta \supseteq \phi$ is a factor $\mathcal{K}$-congruence of A if and only if θ/ϕ is a factor $\mathcal{K}$-congruence of A/ϕ.

Lemma

Let $A \in \mathcal{K}$. If A satisfies $\mathscr{P}$, then so does A/ϕ for every factor $\mathcal{K}$-congruence ϕ of A.

Lemma

Let $A \in \mathcal{K}$ satisfy the condition $\mathscr{P}$. Let $\theta_1 \subseteq \theta_2$ be factor $\mathcal{K}$-congruences of A. If $A \cong A/\theta_2$, then $A \cong A/\theta_1$.

Proof: We construct the sequence $\theta_0 \subseteq \theta_1 \subseteq \theta_2 \subseteq \theta_3 \subseteq \dots$ of factor $\mathcal{K}$-congruences of A so that $A/\theta_n \cong A/\theta_{n+2}$ for all $n \in \mathbb{N}_0$:

- $\theta_0 := \Delta_A$ and $\theta_1 \subseteq \theta_2$ are the initial congruences;
- Once $\theta_0 \subseteq \theta_1 \subseteq \dots \subseteq \theta_{n-1}$ $(n \geq 3)$ satisfying $A/\theta_i \cong A/\theta_{i+2}$ for all $i = 0, 1, \dots, n-3$ are given, the congruence θ_n is defined by the rule

$$\theta_n/\theta_{n-1} = f(\theta_{n-2}/\theta_{n-3})$$

where $f\colon A/\theta_{n-3} \cong A/\theta_{n-1}$.

Skipping trivialities, we have $\theta_0 \subset \theta_1 \subset \cdots \subset \theta_{n-1} \subset \theta_n \subset \dots$.

For every $n \in \mathbb{N}_0$, let ϕ_n/θ_n be the complement $(\theta_{n+1}/\theta_n)^*$ of θ_{n+1}/θ_n in the lattice $\mathrm{Con}_{\mathcal{K}}(A/\theta_n)$. Then ϕ_n is a factor $\mathcal{K}$-congruence of A. Under the isomorphism $A/\theta_n \cong A/\theta_{n+2}$, ϕ_n/θ_n corresponds to ϕ_{n+2}/θ_{n+2}. Hence

$$A/\phi_n \cong (A/\theta_n)/(\phi_n/\theta_n) \cong (A/\theta_{n+2})/(\phi_{n+2}/\theta_{n+2}) \cong A/\phi_{n+2}.$$

It is easily seen that $\phi_j \circ \phi_k = \nabla_A$ for all $j \neq k$. Now, the property $\mathscr{P}$ implies that $\phi_\infty := \bigcap_{n \in \mathbb{N}_0} \phi_n$ is a factor $\mathcal{K}$-congruence of A and

$$A/\phi_\infty \cong \prod_{n \in \mathbb{N}_0} A/\phi_n \cong A/\phi_0 \times A/\phi_1 \times A/\phi_0 \times A/\phi_1 \times \dots,$$

whence

$$A \cong A/\phi_\infty^* \times A/\phi_\infty \cong A/\phi_\infty^* \times A/\phi_0 \times A/\phi_1 \times A/\phi_0 \times A/\phi_1 \times \dots.$$

For every $n \in \mathbb{N}$, ϕ_n/θ_1 is a factor $\mathcal{K}$-congruence of A/θ_1 since $\phi_n \supseteq \theta_n \supseteq \theta_1$. We have $(\phi_j/\theta_1) \circ (\phi_k/\theta_1) = \nabla_{A/\theta_1}$ for $j \neq k$. Since A/θ_1 fulfils $\mathscr{P}$,

$$\psi/\theta_1 := \bigcap_{n\in\mathbb{N}} \phi_n/\theta_1$$

is a factor $\mathcal{K}$-congruence of A/θ_1 and

$$A/\theta_1 \cong (A/\theta_1)/(\psi/\theta_1)^* \times \prod_{n\in\mathbb{N}} (A/\theta_1)/(\phi_n/\theta_1).$$

Obviously, $\psi = \bigcap_{n\in\mathbb{N}} \phi_n$ and so $\phi_\infty = \psi \cap \phi_0$, where $\phi_0 = \theta_1^*$ as $\phi_0/\theta_0 = (\theta_1/\theta_0)^*$ in $\mathrm{Con}_{\mathcal{K}}(A/\theta_0)$ and $\theta_0 = \Delta_A$. Further, let

$$\psi^\natural/\theta_1 := (\psi/\theta_1)^*.$$

Then

$$A/\theta_1 \cong A/\psi^\natural \times \prod_{n\in\mathbb{N}} A/\phi_n.$$

Cantor-Bernstein theorem

Since $\psi^\natural$ is the complement of ψ in $[\theta_1, \nabla_A]_{\mathrm{Con}_\mathcal{K}(A)}$, we have $\psi^\natural = \psi^* \vee \theta_1 = \psi^* \vee \phi_0^* = (\psi \cap \phi_0)^* = \phi_\infty^*$ where ψ^* is the complement of ψ in $\mathrm{Con}_\mathcal{K}(A)$.
Hence

$$\begin{aligned} A/\theta_1 &\cong A/\psi^\natural \times \prod_{n\in\mathbb{N}} A/\phi_n = A/\phi_\infty^* \times \prod_{n\in\mathbb{N}} A/\phi_n \\ &\cong A/\phi_\infty^* \times A/\phi_1 \times A/\phi_0 \times A/\phi_1 \times A/\phi_0 \times \dots \end{aligned}$$

which together with

$$A \cong A/\phi_\infty^* \times A/\phi_0 \times A/\phi_1 \times A/\phi_0 \times A/\phi_1 \times \dots$$

yields $A \cong A/\theta_1$.

Proof of the theorem:

Let $A \cong B \times C$ and $B \cong A \times D$. Then $A \cong A \times D \times C$.
Let θ_1 and θ_2 be the congruences on A corresponding, respectively, to the projections $p_1 : (a, d, c) \mapsto (a, d)$ and $p_2 : (a, d, c) \mapsto a$.
Then $\theta_1 \subseteq \theta_2$ and $A \cong A/\theta_2$. Hence by the last lemma we have $A \cong A/\theta_1 \cong A \times D \cong B$.

The condition $\mathscr{P}$

If $\{\theta_i \mid i \in I\}$ is a countable set of factor $\mathcal{K}$-congruences with $\theta_i \circ \theta_j = \nabla_A$ for all $i \neq j$, then

1. $\theta_\infty := \bigcap_{i \in I} \theta_i$ is a factor $\mathcal{K}$-congruence,
2. $A/\theta_\infty \cong \prod_{i \in I} A/\theta_i$.

In basic algebras, the factor congruences correspond one-one to the central elements:

The condition $\mathscr{P}$ for basic algebras

If $\{a_i \mid i \in I\}$ is a countable set of central elements such that $a_i \wedge a_j = 0$ for all $i \neq j$, then

1. $a_\infty := \bigvee_{i \in I} a_i$ exists and is a central element,
2. for every $\{x_i \mid i \in I\} \subseteq A$ such that $x_i \leq a_i$ for all $i \in I$, the supremum $\bigvee_{i \in I} x_i$ exists.

A basic algebra is **orthogonally σ-complete** if there exists the supremum $\bigvee X$ of every countable subset X such that $x \wedge y = 0$ for all $x \neq y$.

Theorem

Let A and B be orthogonally σ-complete commutative (or effect) basic algebras. If

- $A \cong [0, a] \subseteq B$ for some $a \in C(B)$ and
- $B \cong [0, b] \subseteq A$ for some $b \in C(A)$,

then $A \cong B$.

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